Exploring the Bellman Forest Problem

A Research Paper

by John W. Ward

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Introductions

Synopsis

Recently, survival television has taught us to go downhill to water and downstream to civilization. But in 1956, Richard Bellman **[2]** asked simply: "What is the best path to follow in order to escape a forest of known dimensions?" Results for a variety of forests have been obtained, but a general solution appears elusive and likely unapproachable. The problem was originally posed for unbounded regions – the infinite strip between parallel lines and the half-plane with known distance from the boundary – but bounded regions have also been of interest. This paper presents the branch of results leading to solutions for regular polygonal forests, as well as some related material. Significant proofs are restated, and in a few cases unpublished or unaccessed results are reconstructed. As an extension, results are examined in relation to isosceles triangles.

(Let us grant that the forest setting seems a bit contrived. A lost hiker has a precise map but no compass or landmarks, yet can follow a course involving precise distances and angle measures. Perhaps instead we could imagine a mobile robotic unit equipped only with proximity sensors which must orient itself within a known space by finding a wall.)

With the exception of the half-plane, which was not investigated for this paper, known optimal escape paths are of three types.

- A broad class of forests known as "fat" are best escaped by a linear path. All regular polygonal forests above the triangle are "fat".
- Another class which might be termed "skinny" are best escaped by following a wishbone-shaped path due to Zalgaller.
- Some isosceles triangles including the equilateral are best escaped by a zigzag path due to Besicovitch.

However, the classification of forests is far from complete. There are forests, for instance, which are not "fat", yet their optimal solutions are linear. And there are unsolved forests for which it is known that none of these three paths is optimal.

Definitions and Notation

Points and Lines

- The distance between points A and B is written $\delta(A, B)$.
- The line segment between points A and B is written AB and its length is written $|AB| = \delta(A, B)$.
- The line through points A and B is written $\overline{AB}$.

Forests

- A *forest* is a closed, convex planar set.
- A forest is *bounded* if there is an upper bound to the distance between pairs of points in the forest, and the least upper bound is called *the diameter*. For bounded forest F, we write $Diam(F) = \sup\{\delta(A, B) : A, B \in F\}$. Since forests are closed, the least upper bound will be achieved and we may speak of the diameter as the maximum separation between points, which will necessarily occur only between boundary points.
- *A diameter* of F is any line segment between maximally separated points of a bounded forest.

Paths

- A *path* is a continuous and rectifiable planar arc.
  - The path with endpoints A and B is written $\widetilde{AB}$. Additional points in sequence may be specified within the path, e.g. $\widetilde{ACXDB}$.
  - Path lengths are written $\ell(P)$, $\ell(\widetilde{AB})$, etc.
- Point M is the "midpoint" of path $\widetilde{AB}$ if M lies on $\widetilde{AB}$ and $\ell(\widetilde{AM}) = \ell(\widetilde{MB})$.
- Path $\widetilde{AB}$ is "convex", distinguished by writing $\widehat{AB}$, if adding the line segment BA to form $\widetilde{ABA}$ encloses a convex region. Alternately, a simple path is convex if it lies on its convex hull.
- A "diameter path" for a bounded forest is a line segment of length equal to the forest diameter.

Covers and Escapes

- Forest F "covers" path P if it contains a congruent, orientation-preserving copy of P. Alternately, F covers P if P can be made to fit into F by translation and/or rotation.
- A forest is called an "L-cover" if it covers every path of length L.
- Path P is an "escape path" for forest F if the interior of F does not cover P. Alternately, P escapes F if P cannot be placed in F without intersecting F's boundary.
- If forest F has an escape path, its "escape length" is the greatest lower bound of escape path lengths, written $\ell(F) = \inf\{\ell(P) \mid P \text{ escapes } F\}$, and an escape path P for forest F is "optimal" if $\ell(P) = \ell(F)$.
  - This definition does permit multiple optimal escape paths, and at least one such case exists.
  - This choice for optimality is arbitrary. Other possible definitions, such as minimum expected length, generally have not been investigated.
  - Note that the plane is a forest without escape paths.

The Worm Problem

In 1966, Moser **[11]** posed a related problem which roughly asks: “What's the best-shaped hammer for smashing one-inch worms?” More carefully, Moser asks for the convex shape of smallest area which covers all planar arcs of unit length. Here a planar arc of unit length is called a *worm*, and a planar region which is a unit-cover is also called a *worm cover*. As of 2005, the minimum area known for a convex worm-cover is 0.27381, though a non-convex cover of area 0.26044 is given in **[18]**. Worm problem results are sometimes useful in establishing lower bounds for escape lengths and thereby proving an escape path to be optimal.

Understanding the Problem – The unit square

Before we tackle a problem, we should make sure that we understand its definition and have some sense of how it works. So let's suppose we are lost in a square forest one unit wide. If we know where we are and have a compass handy, we can find the shortest path to an edge (Figure 1).

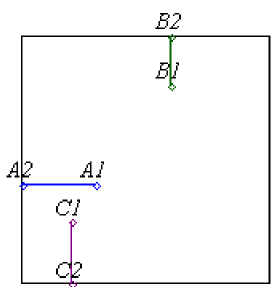


Figure 1

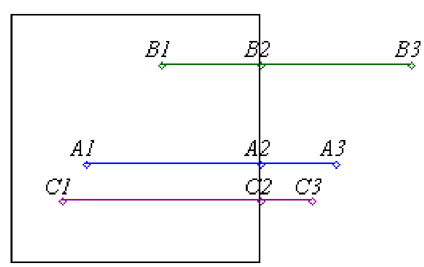


Figure 2

If we have no knowledge of our location but do have a compass, we can walk parallel to one of the sides and and be guaranteed an exit path no longer than one unit, even if it turns out we walked away from the nearest side (Figure 2).

But without a compass, a straight unit walk is not an escape path because we might be walking at an angle to the forest boundary (Figure 3). This is the situation in the Bellman Forest Problem.

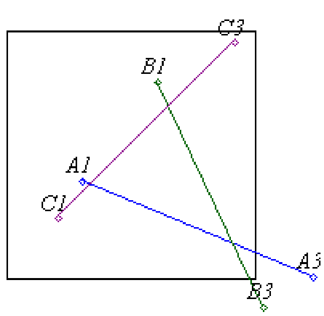


Figure 3

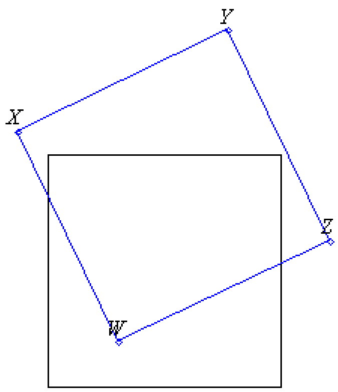


Figure 4

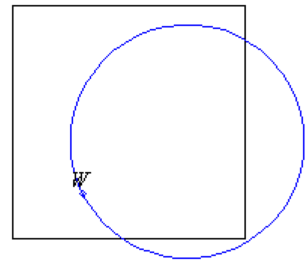


Figure 5

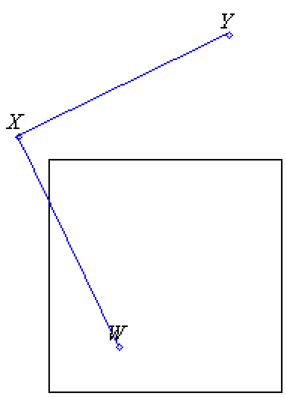


Figure 6

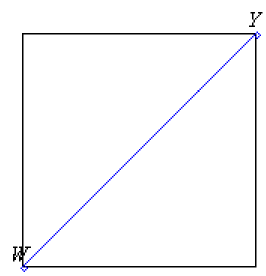


Figure 7

Here are some successful escape paths from a unit square:

- (Figure 4) A unit square: It would be a very strange closed set whose boundary could be contained entirely in its interior. A half-plane might do the trick, but in general a bounded region won't fit inside itself. So we have an escape path of length four – the perimeter of the unit square. The plan would be to walk one unit forward, turn 90 degrees right, walk one unit, and so on.
- (Figure 5) A circle with diameter one: This will also work, and has a length of $\pi \approx 3.1416$, an improvement over the square. The ambitious reader might consider circles of greater diameter, then find required length as a function of diameter and look for a minimum. Mathematicians, of course, ignore the difficulty of walking in a perfect circle, just as we've ignored the problem of bumping into trees.
- (Figure 6) Sides of a square: Someone might note that the fourth side of the square will never be needed to effect an escape, so we have a solution of length 3, another improvement. More consideration might reveal that no more than two sides are actually needed, for an escape length of 2.
- (Figure 7) A straight line (segment): Certainly a segment of length 10 would be an escape path, but a little consideration shows that the square cannot contain a segment of length greater than $\sqrt{2} \approx 1.414$. This is the nicest result so far.

We might be willing to conjecture that the straight line is the best escape path from a square forest. We then face two challenges:

- Can we prove that our conjecture is correct?
- And, if so, can it be generalized to handle other forests?

A note about escape paths

Frequently we will see escape paths placed so as to contact the boundary of the forest more than once. For example, in Figure 8, the escape path $\widehat{W\ X\ Y\ Z}$ begins on the boundary and contacts it twice more before its endpoint. However, if the final leg of the path were any shorter (Figure 9), the path could be rotated and translated slightly such that $\widehat{W\ X\ Y\ Z'}$ would no longer "escape" the forest (Figure 10). Thus it is typical that escape paths are positioned having multiple intersections with the forest boundary.

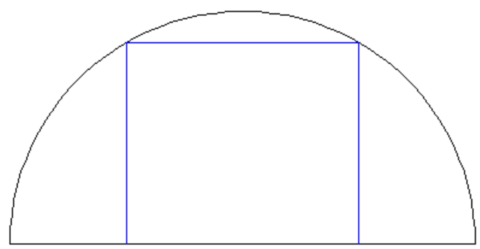


Figure 8

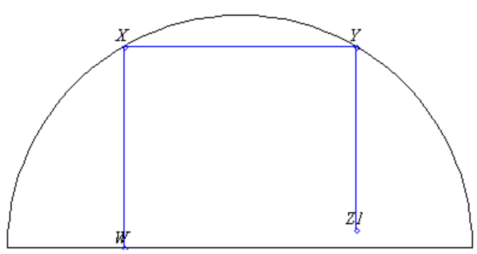


Figure 9

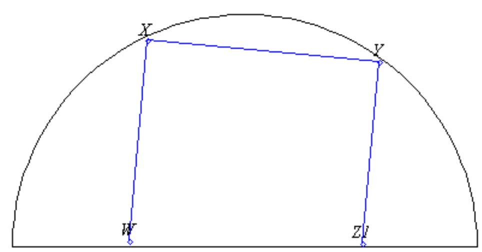


Figure 10

General Arguments

We begin with some basic arguments which will be of repeated use.

Proposition 1: For any bounded forest *F*, a diameter path is an escape path.

If a line segment *AB* with $|AB| = Diam(F)$ is not an escape path, then the interior of *F* covers *AB*. That is, a congruent copy $A'B' \cong AB$ exists with endpoints A' and B' in the interior of *F*. But then there are open $\varepsilon-$neighborhoods of A' and B' lying wholly in the interior of *F*, and so there must exist points A'' and B'' in the interior with $\delta(A'', B'') = Diam(F)+2\varepsilon$, a contradiction. Therefore a diameter path is always an escape from a bounded forest.

Proposition 2 ***(The Escape Bound)***: The length of an escape path is an upper bound for the escape length of that forest. That is, if path *P* escapes forest *F*, $\ell(F) \leq \ell(P)$. In particular, if *F* is bounded, $\ell(F) \leq Diam(F)$.

Proposition 3 ***(The Cover Argument)***: If forest *F* is an *L*-cover, $\ell(F) \geq L$, and so an escape path of length *L* is optimal.

If *F* covers paths of length *L*, its interior covers shorter paths. So *L* is a lower bound for the length of escape paths, and an escape path of length *L* is optimal.

Proposition 4 ***(The Embedding Argument)***: If forest *F* is contained in forest *G*, then $\ell(F) \leq \ell(G)$. That is, the escape length of *F* is a lower bound for the escape length of *G*, and the escape length of *G* is an upper bound for the escape length of *F*.

Corollary 5: A forest containing an *L*-cover is also an *L*-cover.

Linear Solutions

The Circular Disk (Gross, 1955)

Since the Forest Problem permits any continuous path, it is not obvious where to begin. However, in 1955, O. Gross, got the ball rolling. In *A Search Problem Due to Bellman* **[7]**, he concluded that the optimal escape path from a circular disk is a line segment equal in length to the diameter. It is relatively easy to show that the line segment is an optimal escape path. Then it is intuitively obvious, but not quite so easy, to show that it is uniquely optimal.

Proposition 6: A diameter path is the optimal escape from a circular disk.

First we show that a circular disk covers all arcs of diameter length. Let F be a forest bounded by a circle of radius r, and P a path. Let M be the midpoint of P and place M on the center of the circle. If P intersects the circle at some point A, then $\delta(M, A) = r$. Since A lies on one semi-path of P, the length of the semi-path containing A is at least r, and the length of P must be at least $2r$, the diameter of the circle. Therefore F covers all paths of diameter length, and so $Diam(F) \leq \ell(F)$. But since a diameter path is an escape path of diameter length, it must be optimal.

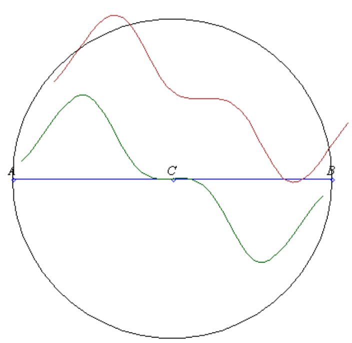


Figure 11

Next we show that a diameter path is uniquely optimal. Let $\widehat{AMB}$ be any path with $\ell(\widehat{AMB}) = Diam(F)$ and midpoint M. If $\widehat{AM}$ and $\widehat{MB}$ are non-collinear line segments of length r, we can easily cover $\widehat{AMB}$ with the interior of F. Simply place M at the center of the circle and then move it along the bisector of $\measuredangle AMB$ toward the major arc defined by AB.

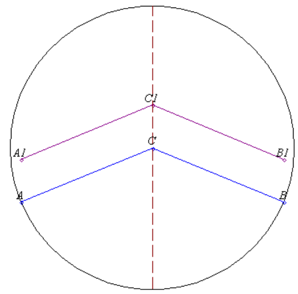


Figure 12

If one of the semi-paths, say $\widehat{AM}$ is not linear, then for every point $p \in \widehat{AM}$, $\delta(p, M) < r$. Then we can find a point $M' \neq M$ on $\widehat{MB}$ such that $\delta(p, M') < r \;\; \forall\, p \in \widehat{AM'}$. If we place M' at the center of the circle, $\widehat{AM'}$ cannot intersect the circle since none of its points are a distance of *r* from the center, and $\widehat{M'B}$ cannot intersect the circle since now $\ell(\widehat{M'B}) < r$. Thus a line segment is the unique optimal escape from a circular forest.

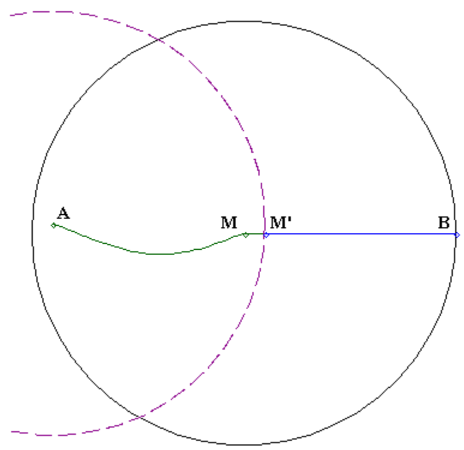


Figure 13

The "Fat" Forest (Gerriets and Poole, 1973)

Can similar reasoning determine other forests with optimal escape paths which are linear? In 1973, George Poole and John Gerriets published a brief note **[13]** regarding worm covers. This was followed the next year by a more thorough explanation **[6]**.

The result is pleasantly straightforward, both in statement and proof. In the authors' words, "We must admit that even though the result is rather striking the proof is quite simple." (**[6]**, p. 37) Yet simplicity does not imply that the result was obvious, so if only for the discovery Gerriets and Poole certainly deserve their credit. Their argument in **[6]** is so concise that it is reproduced in its entirety along with the original accompanying figure. Where it seems useful for clarification we add a few footnotes.

Proposition 7: (Theorem 1 in Gerriets and Poole)

> *The closed region whose boundary is a rhombus with major diagonal L and minor diagonal $L/\sqrt{3}$ covers any arc of length L.*

> *Proof.* Let α be any arc of length *L* with midpoint *O* which divides α into two subarcs β and γ. *ABCD* will denote the rhombus described in the theorem with *BD* having length $L/\sqrt{3}$. Suppose first that there is an orientation of α with *O* on *BD*, all points of α on or above angle *ABC*, and both β and γ contiguous with angle *ABC*.(1) Suppose also that β meets *BC* at point *P* and meets *DC* at point *Q* with *P* lying between *O* and *Q*.(2) Construct *BS* and *PR* perpendicular to *DC* and construct *PT* and *PV* perpendicular to *BS* and *BD*, respectively. Then the length of $\beta \geq OP + PQ \geq VP + PR = BT + TS = L/2$ which shows that β cannot cross *DC*.(3) If *Q* lies between *O* and *P*, the argument is similar by symmetry. Similarly, γ is covered by the rhombus in this case.

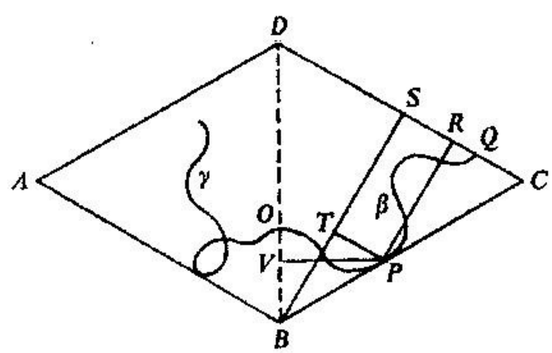


FIG. 1.

*Figure 14: From **[6]***

> Secondly, suppose in all possible orientations of α with *O* on *BD* as described above that only one arc, say β, is contiguous with angle *ABC*.(4) Therefore, in any of these positions, as argued above, the rhombus covers β and by symmetry of the rhombus and the assumption that γ cannot touch angle *ABC*, γ cannot pass through angle *ADC*. For suppose γ even so much as touches angle *ADC*, then there is an orientation where it also touches angle *ABC*

(since the figure is symmetric about line *AC*), contrary to the conditions of this case. Thus the rhombus covers all arcs of length *L*.

Notes (in case the above was not all "intuitively obvious to the casual observer"):

(1) Clearly it is possible to position α above $\measuredangle ABC$ with *O* on *BD,* and at least one of β and γ contiguous with the angle. This does not require that the subarcs be contiguous with segments *AB* and *BC*, merely the angle they describe. It seems most likely that some orientation will place both subarcs contiguous with the angle. However, a simple counterexample would be to have γ lie entirely in the convex hull of α.

(2) Since β lies entirely above $\measuredangle ABC$, if it intersects *BC* but not *DC* it cannot escape from the right side of the rhombus. And if β contacts both segments it must do so in some order. In the case where β intersects both segments at *C*, clearly β has length at least $L/2$.

(3) $\beta \geq OP+PQ \geq VP+PR = BT+TS = L/2$: OP has greater length than the perpendicular *VP*, and likewise $PQ \geq PR$. Clearly $RP = ST$. Since $m\measuredangle VBP = m\measuredangle BCS = 60^\circ$, $m\measuredangle BPV = m\measuredangle CBS = 30^\circ$, so $\Delta BPV \cong \Delta PBT \Rightarrow PV = BT$.

Gerriets and Poole do not discuss the possibility that β intersects *BC* and then exits the left side of the rhombus. However, this is not a problem. Since the arc lies above $\measuredangle ABC$, to exit the rhombus it would be necessary to intersect *AD*, and all points on *BC* are at least $L/2$ away from *AD*.

(4) It would appear the authors are only addressing the situation where one particular subarc, say γ, never intersects $\measuredangle ABC$, perhaps leaving themselves open to a counterexample in which some orientations have β intersecting $\measuredangle ABC$ and some have γ intersecting $\measuredangle ABC$, but no orientations where both subarcs intersect the angle. The situation, however, will not arise. For each rotation of α with *O* on *BD*, we can slide *O* along *BD* until something contacts $\measuredangle ABC$ but nothing passes below it. Suppose this is a point *X* on β. As we rotate α counterclockwise from this orientation, *X* will remain contiguous with line *BD* until some other point of the arc intersects the angle to the left of *X* and continued rotation forces *X* off the angle. (This follows the *Wingwalker's Rule*: "Never let go of what you got ahold of 'til you got ahold of something else.") Thus if β intersects the angle in some rotations and γ in others, there must exist a transitional orientation where both intersect the angle.

Notice that in none of this do the authors discuss escape paths, though their proof clearly demonstrates an escape path in order to achieve a contradiction. All they have claimed is that the rhombus is an *L*-cover, which we recognize to mean that no escape path may have length less than *L*. However, since a line segment of length *L* is a diameter path, it is indeed an escape path.

- Implications -

The utility of this theorem relies on the embedding argument. Any forest containing such a rhombus is clearly also an *L*-cover, and, if the forest has diameter *L*, then a diameter path is an optimal escape.

Definition (*The Fat Forest*):

> "Call a compact, convex set *X fat* if it contains points *P* and *Q* so that (a) *PQ* is the diameter of *X* and (b) the 60° rhombus *R(PQ)* with longer diagonal *PQ* fits in *X*." (Finch & Wetzel **[5]**, p. 647)

Proposition 8: "The escape length of a fat forest is its diameter." (**[5]**, p. 647) The conclusion is immediate – the fat forest covers all arcs of diameter length, so a diameter path is optimal.

Fortunately, quite a number of forests are *fat*. Most notably, every regular polygonal forest with at least four sides is fat and is therefore best escaped by a straight path. In particular, this establishes our conjecture for the unit square. Circles are fat, but we already had a proof regarding them.

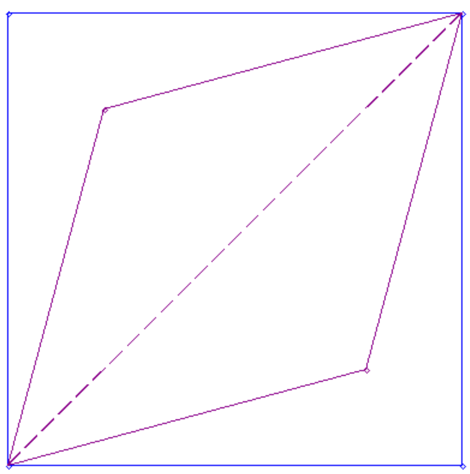


Figure 15

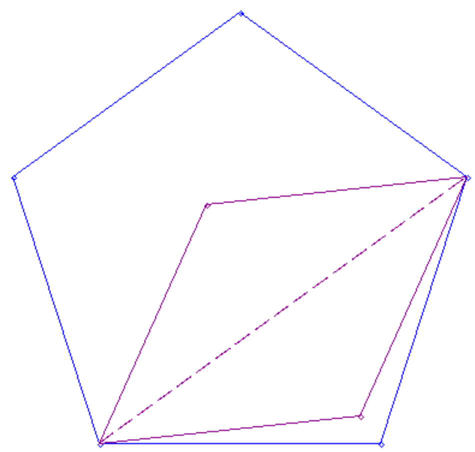
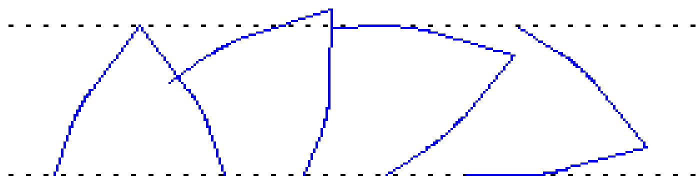


Figure 16

The Zalgaller Solution

Minimum Path of Unit Width (Zalgaller, 1961)



Part of Bellman's initial problem asked for an escape path from an infinite strip – two parallel lines at distance *w* and the planar region between them. Clearly a straight walk is no use. Two perpendicular legs, each of length $w\sqrt{2}$, will find a border after a walk of no more than $2.82843\,w$. Even better, two legs of an equilateral triangle with height *w* will succeed in no more than $2.30941\,w$. What would an optimal path look like?

> To the reader: This discussion of the Zalgaller path is, for the most part, a restatement of the proof found in Adhikari and Pitman **[1]**. Where other works are used they will be cited. Attempts to simplify or generalize their argument will be noted.

Definition: For a path *P*, define $w_\theta(P)$ as the "distance between supporting parallel lines at an angle of θ to the *x*-axis". Define the "path width**"** by $w(a) = \inf_{0 \le \theta \le \pi} w_\theta(a)$. In other words, the width of a path is the minimum separation between parallel lines capable of enclosing the path. Similarly, for any bounded forest *F* we define the *forest width* $w(F)$ as the minimum distance between parallel lines tangent to the forest. It is clear that the width of a path or forest is the same as the width of its convex hull.

For simplicity, we choose our forest as the region between lines one unit apart, so our goal is to find the shortest path of unit width. According to Finch & Wetzel **[5],**

> "...the solution was described in 1961 by V. A. Zalgaller....It was rediscovered in 1968 by Schaer, who called it the *broadworm* and provided a careful proof of its uniqueness; it was rediscovered again in 1986-87 by Klötzler and Klötzler and Pickenhain, who called it the *universal escape path*; and it was rediscovered yet again in 1989 by Adhikari and Pitman, who called it the *caliper*."

The Zalgaller path $\mathcal{Z}$ appears in Figure 17. Its dimensions are given in Wetzel **[17]** based on Schaer **[14]**:

- Define the critical angles $\alpha = \arcsin\left[\frac{1}{6} + \frac{4}{3}\sin\left(\frac{1}{3}\arcsin\frac{17}{64}\right)\right] \approx 0.290046$,

  $\gamma = \arctan\left(\frac{1}{2}\sec\alpha\right) \approx 0.480931$, and $\beta = \frac{\pi}{2} - \alpha - 2\gamma \approx 0.318888$.
- The segment AB, not part of the path, has a length of $\sec\alpha \approx 1.043590$.

- The right side of the path $\widehat{RSTB}$ is composed of arc ST and segments RS and TB, where R is one unit above the midpoint of $\overline{AB}$ and ST is a circular arc of radius 1 centered at A. The left side $\widehat{APQR}$ is symmetric to the right side.
- The path has a total length of $\ell(\mathcal{Z}) = \zeta = 2(\tan\alpha + \beta + \tan\gamma) \approx 2.278291644$, just slightly better than our result based on the equilateral triangle.

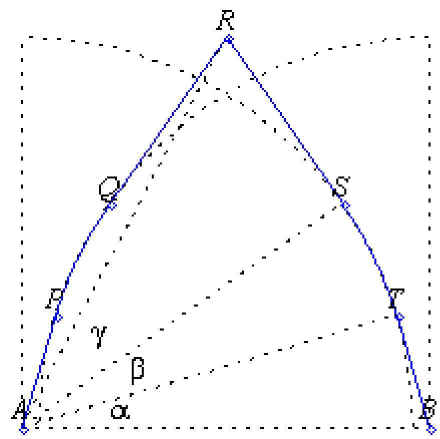


Figure 17: The Zalgaller path

Proposition 9: The Zalgaller path is the shortest path of unit width.

To show that Zalgaller is optimal, we first establish that $\mathcal{Z}$ is the shortest *convex* path of unit width. Then we argue that it is the shortest among all paths.

Proposition 9.1: The Zalgaller path is the shortest convex path of unit width.

Suppose arc $\widehat{AB}$ is the shortest convex arc of unit width. Since it is convex, the entire arc must lie on one side of the line $\overline{AB}$. Without loss of generality, we can orient the arc in the first quadrant of an *xy*-coordinate system such that the endpoints are on the *x*-axis with point $A = (a, 0)$ to the left of $B = (b, 0)$, and the arc $\widehat{AB}$ tangent to the *y*-axis at $Q = (0, c)$. Let *w* denote greatest *x*-coordinate of any point on the arc with a tangent point $R = (w, d)$, and let *h* be the height, or greatest *y*-coordinate, of the arc. Since the arc has width at least one, $w \geq 1$ and $h \geq 1$ (Figure 18).

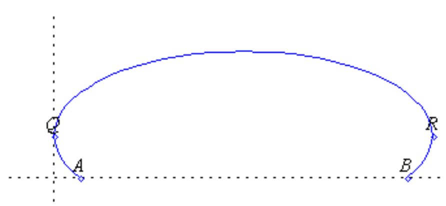


Figure 18

Clearly, if $a > 0$, we could replace the arc $\widehat{AQ}$ with vertical line segment $A'Q$ where $A' = (0, 0)$ to obtain a shorter convex arc of the same width. Similarly, if $b < w$, we could obtain a shorter arc with $B' = (w, 0)$. So we may conclude that in the shortest convex arc, $A = (0, 0)$

and $B = (w, 0)$.

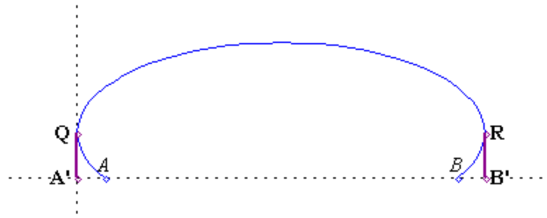


Figure 19

Next we consider a construction with point *A* at the origin, point *B* at $(w, 0)$ for some $w \geq 1$, a horizontal line $A'B'$ at unit height above *AB*, and open discs *Disc(A)* and *Disc(B)* of unit radius centered at *A* and *B* respectively (Figure 20).

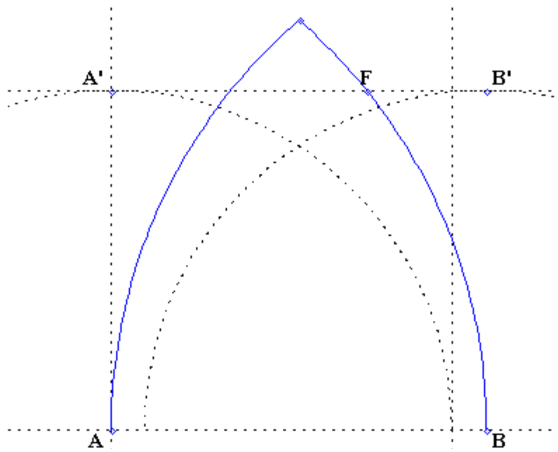


Figure 20

We claim that paths so aligned have the following characteristics: (**[1],** p. 312)

"Lemma 1 : *For a convex arc* $\widehat{AB}$ *above* AB *with* $|AB| \geq 1$, $width\left(\widehat{AB}\right) \geq 1$ *if and only if the following three conditions all hold:*

1. $\widehat{AB}$ *intersects* $A'B'$ *at some point F.*
2. $\widehat{AF}$ *does not intersect Disc(B).*
3. $\widehat{FB}$ *does not intersect Disc(A).*"

$\Rightarrow$ Suppose the three conditions hold. To show that the arc has width at least one, it is necessary to show that the arc cannot be contained strictly between parallel lines one unit apart. Our construction already assumes horizontal width of at least one unit, and the first condition guarantees a vertical unit width.

Let L_1 and L_2 be down-sloping parallel lines, one unit apart, with L_1 the upper line. Since the path does not extend below or left of point *A*, we may translate the path so that *A* lies on L_2, which will place L_1 tangent to the unit circle centered at *A*. If L_1 passes between *A* and *B*, the continuity of $\widehat{AB}$ guarantees that it intersects L_1. Thus we only need to consider the case where L_1 intersects $\overline{AB}$ to the right of *B* (Figure 21).

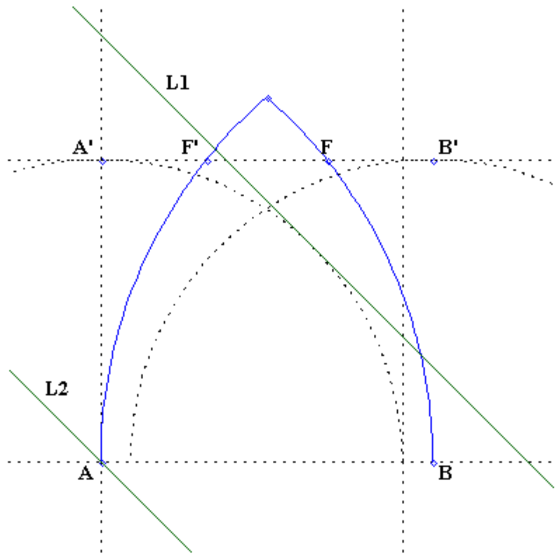


Figure 21

If F lies on or to the right of L_1, then $\widehat{FB}$ clearly intersects L_1. On the other hand, if F lies to the left of L_1 (labeled F' in the figure), then $\widehat{F'B}$ cannot remain below L_1 without intersecting open *Disc(A)*, and so must intersect L_1. Thus $\widehat{AB}$ has at least unit width measured by down-sloping lines. The same can be shown for up-sloping lines by aligning them with point B and *Disc(B)*. Therefore, when the three conditions hold, the path has unit width.

$\Leftarrow$ Suppose $\widehat{AB}$ has unit width. Clearly the path must intersect the line one unit above $\overline{AB}$, namely $\overline{A'B'}$, so the first condition holds. Assume, for the sake of contradiction, that $\widehat{AF}$ intersects *Disc(B)*. Since A and F cannot lie in open *Disc(B)*, there must exist points C and D lying on the unit arc centered at B such that $\widehat{AF} = \widehat{ACDF}$ and all points other than C and D of $\widehat{CD}$ lie within *Disc(B)*. Then, to preserve convexity, the line segment CD must lie in the interior of the convex hull of $\widehat{AB}$.

Now consider line L_1 parallel to $\overline{CD}$ and tangent at point E to the unit circle centered at B. To maintain unit width, $\widehat{AF}$ must contain some point G on L_1. Since L_1 lies outside open *Disc(B)*, G is not on $\widehat{CD}$, and so must lie on $\widehat{AC}$ or $\widehat{DF}$. Suppose G is on $\widehat{DF}$, as in Figure 22. Then to preserve convexity, the line segment CG must intersect the unit circle centered at B somewhere between C and D, and that point must lie within the convex hull of $\widehat{AB}$, which can only reasonably occur if the intersection lies on $\widehat{CD}$, contradicting our assumption. A corresponding problem arises if G is on $\widehat{AC}$. Thus $\widehat{AF}$ cannot intersect *Disc(B)*, and so the three conditions hold.

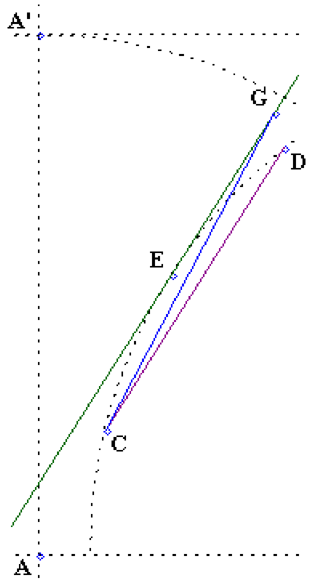


Figure 22

To find the optimal path satisfying these conditions, first we fix some $d = |AB|$ and some point F on $A'B'$. If the path $\widehat{AFB}$ has unit width, by the lemma, $\widehat{AF}$ does not intersect *Disc(B)* and $\widehat{FB}$ does not intersect *Disc(A)*. If line segments *AF* or *FB* do not intersect *Disc(B)* or *Disc(A)*, respectively, they are the shortest choices for that portion of the path. The minimum base width *d* at which two line segments can be used occurs when $d = \frac{2}{\sqrt{3}}$ and *F* is the midpoint of $A'B'$, with $\ell(\widehat{AFB}) = 2\sqrt{1^2+\left(\frac{d}{2}\right)^2} = 2\sqrt{\frac{4}{3}} = 4\frac{\sqrt{3}}{3} \approx 2.3094$. (Figure 23.) Clearly, when $d > \frac{2}{\sqrt{3}}$, the path will be no shorter, so we need only consider values where $1 \le d \le \frac{2}{\sqrt{3}}$.

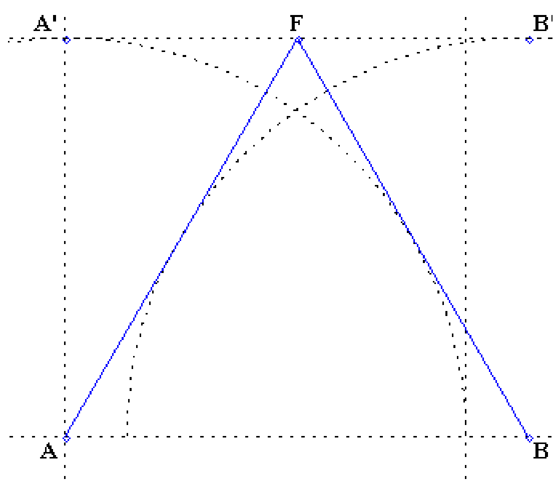


Figure 23

For $d < \frac{2}{\sqrt{3}}$, it will be necessary for at least one of $\widehat{AF}$ and $\widehat{FB}$ to be non-linear in order to avoid intersecting the open discs. This is best accomplished by following a portion of the unit arc surrounding the disc, then using line segments to minimize the remaining distance. Convexity requires that these line segments lie on lines tangent to the discs.

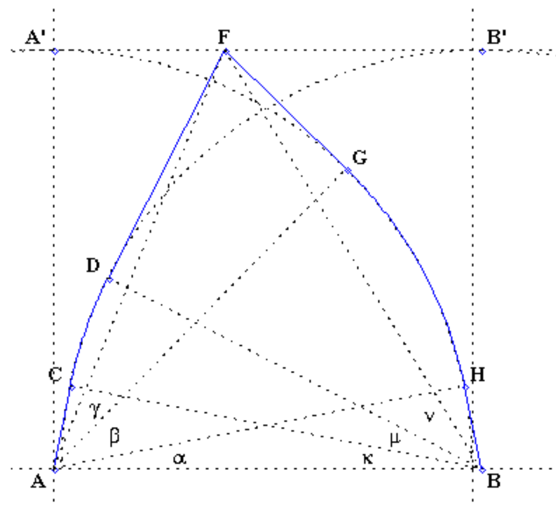


Figure 24

We can calculate the angles involved and therefore the path length. Let $F = (x_F, 1)$ for some $0 \le x_F \le d$, and construct a candidate path $\mathcal{Z}^*$ as in Figure 24. Since $|BC| = |AH| = 1$, $\alpha = \kappa = \cos^{-1}\left(\frac{1}{d}\right)$. Similarly, since $|BD| = |AG| = 1$, $\nu = \cos^{-1}\left(\frac{1}{|FB|}\right) = \cos^{-1}\frac{1}{\sqrt{1+(d-x_F)^2}}$ and $\gamma = \cos^{-1}\left(\frac{1}{|AF|}\right) = \cos^{-1}\frac{1}{\sqrt{1+x_F^2}}$. Then since $\tan(\sphericalangle BAF) = \frac{1}{d-x_F}$, $\beta = \tan^{-1}\left(\frac{1}{d-x_F}\right) - \gamma - \alpha$, and similarly $\mu = \tan^{-1}\left(\frac{1}{x_F}\right) - \nu - \kappa$.

We can then compute the path length. Since $|AF| = \sqrt{1+x_F^2}$, $|DF| = \sqrt{|AF|^2 - 1} = \sqrt{x_F^2} = x_F$, and similarly, since $|FB| = \sqrt{1+(d-x_F)^2}$, $|FG| = \sqrt{|FB|^2 - 1} = \sqrt{(d-x_F)^2} = d - x_F$. Thus, $|DF| + |FG| = d$. Further, $|AC| = |HB| = \sqrt{d^2-1}$. So $\ell(\mathcal{Z}^*) = \zeta^*$ is equal to $2\sqrt{d^2-1} + d + \beta + \mu$.

It seems reasonable that for any given *d*, the optimal choice for *F* is the midpoint of $A'B'$. Adhikari and Pitman argue that reflecting $\widehat{FB}$ about $A'B'$ shows the path is clearly minimized when *DF* and the reflection of *FG* are collinear. We can also approach the proof via calculus.

Digression – a solution via calculus

To minimize $f(X_F)=\zeta^*$ for any given *d*, we must minimize the non-constant value $\beta+\mu$, which can be expressed as a function $(\mu+\beta)(x_F) = \tan^{-1}\frac{1}{x_F} + \tan^{-1}\frac{1}{(d-x_F)} - \cos^{-1}\frac{1}{\sqrt{1+(d-x_F)^2}} - \cos^{-1}\frac{1}{\sqrt{1+x_F^2}} - 2\cos^{-1}\frac{1}{d}$. Differentiating the summands, we get

- $\frac{d}{dx_F}\tan^{-1}\frac{1}{x_F} = \left(\frac{1}{1+\frac{1}{x_F^2}}\right)\left(\frac{-1}{x_F^2}\right) = -\frac{1}{x_F^2+1}$,
- $\frac{d}{dx_F}\tan^{-1}\frac{1}{d-x_F} = \left(\frac{1}{1+\frac{1}{(d-x_F)^2}}\right)\left(\frac{-1}{(d-x_F)^2}\right)(-1) = \frac{1}{(d-x_F)^2+1}$,
- $$\begin{aligned}\frac{d}{dx_F}\left(-\cos^{-1}\frac{1}{\sqrt{1+(d-x_F)^2}}\right) &= \frac{(-1)(-1)}{\sqrt{1-\frac{1}{1-(d-x_F)^2}}}\left(\frac{-1\cdot 2(d-x_F)(-1)}{2\sqrt{1+(d-x_F)^2}(1+(d-x_F)^2)}\right)\\ &= \frac{d-x_F}{\sqrt{1+(d-x_F)^2-1}(1+(d-x_F)^2)},\\ &= \frac{d-x_F}{|d-x_F|(1+(d-x_F)^2)} = \frac{1}{1+(d-x_F)^2}\end{aligned}$$
- $$\begin{aligned}\frac{d}{dx_F}\left(-\cos^{-1}\frac{1}{\sqrt{1+x_F^2}}\right) &= \frac{(-1)(-1)}{\sqrt{1-\frac{1}{1-x_F^2}}}\left(\frac{-1\cdot 2x_F}{2\sqrt{1+x_F^2}(1+x_F^2)}\right)\\ &= -\frac{x_F}{\sqrt{1+x_F^2-1}(1+x_F^2)},\\ &= \frac{-1}{1+x_F^2}\end{aligned}$$
- and $\frac{d}{dx_F}\cos^{-1}\frac{1}{d} = 0$.

Thus $\frac{d}{dx_F}(\beta+\mu)(x_F) = \frac{2}{1+(d-x_F)^2} - \frac{2}{1+x_F^2}$. We note immediately that when $x_F = \frac{d}{2}$, $x_F = d-x_F$, and $\frac{d}{dx_F}(\beta+\mu) = 0$. Since $\frac{d^2}{dx_F^2}(\beta+\mu)(x_F) = \frac{-2\cdot 2(d-x_F)(-1)}{(1+(d-x_F)^2)^2} - \frac{-2\cdot 2x_F}{(1+x_F^2)^2} = \frac{4(d-x_F)}{(1+(d-x_F)^2)^2} + \frac{4x_F}{(1+x_F^2)^2} > 0$, our path is optimal when symmetric, and indeed $x_F = \frac{d}{2}$.

Substituting, we get $(\mu+\beta)(d) = 2\left(\tan^{-1}\frac{2}{d} - \cos^{-1}\frac{2}{\sqrt{4+d^2}} - \cos^{-1}\frac{1}{d}\right)$, and so for a given *d*, $g(d) = \ell(\zeta^*) = d + 2\left(\sqrt{d^2-1} + \tan^{-1}\frac{2}{d} - \cos^{-1}\frac{2}{\sqrt{4+d^2}} - \cos^{-1}\frac{1}{d}\right)$. Then

$$
\begin{aligned}
g'(d) &= 1+2\left[\frac{d}{\sqrt{d^2-1}} - \frac{2}{d^2+4} - \frac{2}{d^2+4} - \frac{1}{d\sqrt{d^2-1}}\right] \\
&= 1+\frac{2d}{\sqrt{d^2-1}} - \frac{8}{d^2+4} - \frac{2}{d\sqrt{d^2-1}} \quad . \\
&= 1+\frac{2\sqrt{d^2-1}}{d} - \frac{8}{d^2+4}
\end{aligned}
$$

A numerical solution for $g'(d) = 0$ yields an optimal value at $d \approx 1.043590110$.

Digression – Trigonometric Solution of the Cubic

Instead of the numerical solution, Wetzel **[17]** gives $d = sec(\alpha)$ where $\alpha = \sin^{-1}\left[\frac{1}{6}+\frac{4}{3}\sin\left(\frac{1}{3}\sin^{-1}\frac{17}{64}\right)\right]$, based on "expressions that arise from the trigonometric solution of the cubic that appears in an extremum problem." How does that work?

From Figure 24, is apparent that $\sin\alpha = \frac{\sqrt{d^2-1}}{d}$ and $\cos^2\alpha = 1-\sin^2\alpha = \frac{1}{d}$, so

$g'(d) = 1+2\sin\alpha - \frac{8}{\left(\frac{1}{1-\sin^2\alpha}+4\right)} = \frac{8\sin^3\alpha-4\sin^2\alpha-10\sin\alpha+3}{4\sin^2\alpha-5}$, and we may solve for

$8\sin^3\alpha-4\sin^2\alpha-10\sin\alpha+3 = 0$ or just $\sin^3\alpha-\frac{1}{2}\sin^2\alpha-\frac{5}{4}\sin\alpha+\frac{3}{8} = 0$. We can remove the second-degree term by the substitution $\sin\alpha = y+\frac{1}{6}$, yielding $0 = y^3-\frac{4}{3}y+\frac{34}{216}$. Here's where the trigonometry comes in. Following Lambert **[9]**, we seek the identity $\sin^3\varphi-\frac{3}{4}\sin\varphi+\frac{1}{4}\sin 3\varphi = 0$. By the substitution $y = \frac{4}{3}x$, we get $0 = \frac{64}{27}x^3-\frac{16}{9}x+\frac{17}{108}$, and clearing the leading coefficient gives $0 = x^3-\frac{3}{4}x+\frac{17}{256}$. This will occur when $x = \sin\varphi$ and $\frac{1}{4}\sin 3\varphi = \frac{17}{256}$. So we conclude that $\sin 3\varphi = \frac{17}{64} \Rightarrow$

$3\varphi = \sin^{-1}\frac{17}{64} \Rightarrow \varphi = \frac{1}{3}\sin^{-1}\frac{17}{64} \Rightarrow x = \sin\varphi = \sin\left(\frac{1}{3}\sin^{-1}\frac{17}{64}\right) \Rightarrow$

$y = \frac{4}{3}\sin\left(\frac{1}{3}\sin^{-1}\frac{17}{64}\right) \Rightarrow \sin\alpha = \frac{1}{6}+\frac{4}{3}\sin\left(\frac{1}{3}\sin^{-1}\frac{17}{64}\right) \Rightarrow \alpha = \sin^{-1}\left[\frac{1}{6}+\frac{4}{3}\sin\left(\frac{1}{3}\sin^{-1}\frac{17}{64}\right)\right]$.

Always nice to have rational results, even if we need transcendental functions to get them.

So far we have shown that the Zalgaller path is the shortest convex path of unit width. It remains to be shown that it is shortest among all paths. It may be a bit surprising that we would need to consider non-convex paths, since the utility of a path is based on its convex hull. But as we shall see with Besicovitch, a convex path is not always the shortest way of establishing a convex hull. A thin rhombus should prove sufficient example (Figure 25).

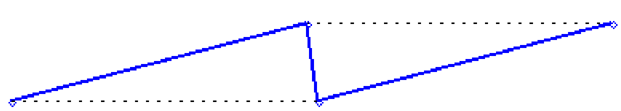


Figure 25

The following reworks the argument in **[1]**, attempting to avoid one of the issues in the original paper. It skips several introductory arguments and reaches their Lemma 5 differently but based on their original construction. Specifically, the authors deal only with paths composed of finite numbers of line segments. They indicate that they are unable to demonstrate their argument for paths with infinite segments, but by minor modifications we will attempt to handle non-linear segments and to eliminate the possibility of infinite "crossings". Our basic goal is also theirs – to show that a minimal arc of width one must contain a "convex arch of unit height", and then that a minimal arc with such an "arch" must indeed be convex.

Let path P be a minimal arc of unit width.

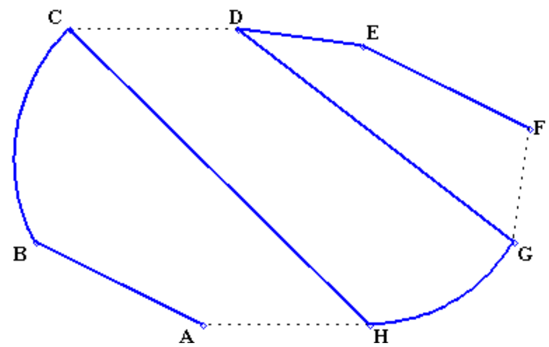


Figure 26

First we make a few observations, hopefully intuitively clear (Figure 26).

- No congruent copy of the convex hull of P can be contained completely in the interior of P.
- There can be no path P' shorter than P whose convex hull is greater than or equal to the convex hull of P by containment, since then P' is an escape path and P is not minimal.
- Path P is divisible into subpaths lying on the convex hull of P connected by "crossings" - i.e., subpaths lying on the interior of the convex hull with the exception of their endpoints, which lie on the hull. Segments CH and GD are crossings in Figure 26.
- Since P is minimal, all its crossings must be line segments.
- Since P is minimal, its endpoints must lie on its convex hull. In fact, if an endpoint lies on a portion of the hull which is a line segment, it must lie on an endpoint of that segment.

It should also be clear that P does not intersect itself. (**[1]**, Lemma 3) If a crossing terminates at a point on the convex hull that is already on the path, it is wasted (as AC in Figure 27). And if two crossings intersect, they can be "uncrossed" to produce a shorter path (Figure 28).

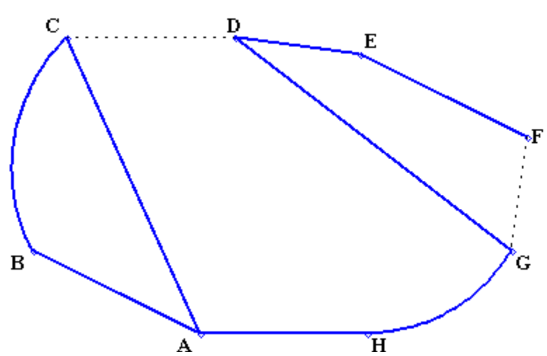


Figure 27

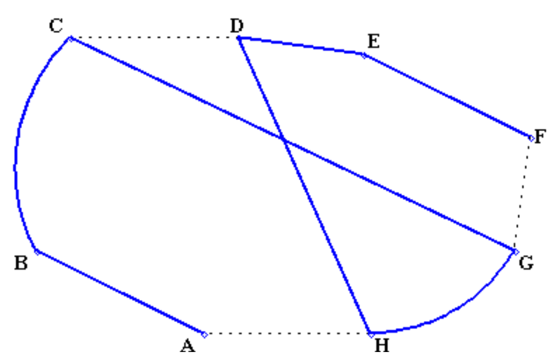


Figure 28

Next, define an "arch" as a continuous section of our escape path which begins and ends either with a "crossing" or at an endpoint of the path, and otherwise follows the convex hull. In Figure 26, $\widehat{ABCH}$, $\widehat{CHGD}$, and $\widehat{GDEF}$ are arches. Crossings only occur at the ends of the arch and do not intersect, so each arch of a path is a convex. Define the "height" of the arch as the greatest distance of a point on the arch from a line passing through its endpoints. Here, the height of $\widehat{ABCH}$ is the distance of point C from line $\overline{AH}$, and the height of $\widehat{GDEF}$ is the distance of point D from $\overline{GF}$.

Proposition 9.2: *A minimal path of unit width has an arch of at least unit height*. (cf. **[1]**, Lemma 5)

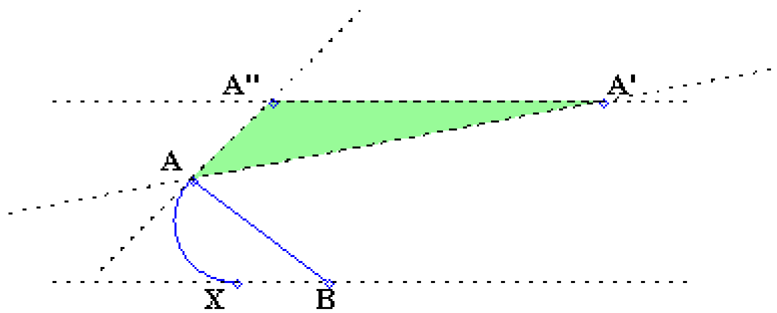


Figure 29

Let the endpoints of path P be labeled X and Y, and consider the arch beginning at X. If this arch is the entire path, it must have at least unit height or fail to have unit width. If, however, the first arch ends with a crossing AB and the height of $\widehat{XAB}$ is less than one, then there must be some point A' on the convex hull representing the highest point on path P above line $\overline{XB}$. The arc $\widehat{AA'}$ of the convex hull must lie on or above line $\overline{AA'}$ to preserve convexity. Further, $\widehat{XA}$ will end at A with either a line segment or a convex curve. In either case, the slope of a line through A will be determined such that $\widehat{AA'}$ must lie on or below this line to preserve convexity. This second line must intersect a line parallel to $\overline{XB}$ through A', and we may label their intersection A''. As a result, the arc $\widehat{AA'}$ of the convex hull must lie in the triangular region determined by A, A', and A'' (Figure 29).

Whether the second arch of path P ends at Y or with a crossing, it must end within that triangular region. Suppose that the second arch ends with the crossing CD and that again the height of the arch is less than one (Figure 30). By the same logic as above, there must be a point D' at maximum distance from $\overline{AD}$, and the arc $\widehat{CD'}$ on the convex hull of P must lie in the triangular region determined by C, D', and D'', where D'' is the intersection of the line through D' parallel to $\overline{AD}$ and the tangent to the path at C.

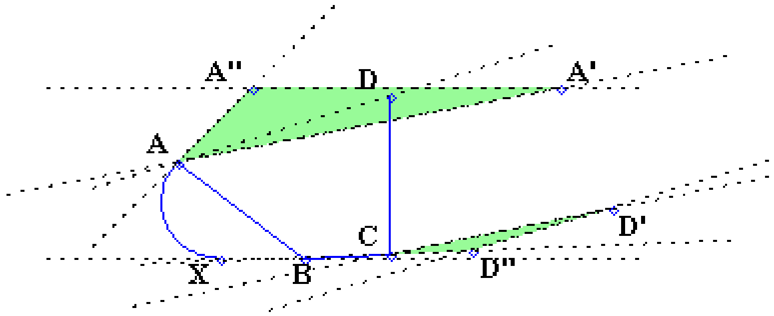


Figure 30

Of necessity, the slope of $\overline{AD} \parallel \overline{D''D'}$ is at least that of $\overline{BC}$ which in turn is at least that of $\overline{XB}$. Since therefore $\overline{AD}$ and $\overline{BC}$ do not converge to the right, the height of arch $\widehat{ABCD}$ (equal to the height of C over $\overline{AD}$), is at least the height of B over $\overline{AD}$, which is at least the height of A over $\overline{BC}$, which finally is at least the height of A over $\overline{XB}$ (Figure 31). (In minimal paths, these should be strict inequalities, but it makes no difference to the argument.) As a result, we conclude that the height of arch $\widehat{ABCD}$ is at least that of arch $\widehat{XAB}$, and further arches must in turn have non-decreasing heights. Since P is by hypothesis a minimal escape path, as long as the convex hull is of finite dimension there cannot be an infinite series of arches. Therefore, P has a finite number of arches, but arches of less than unit height force the path to continue, so we must have at least one arch of unit height.

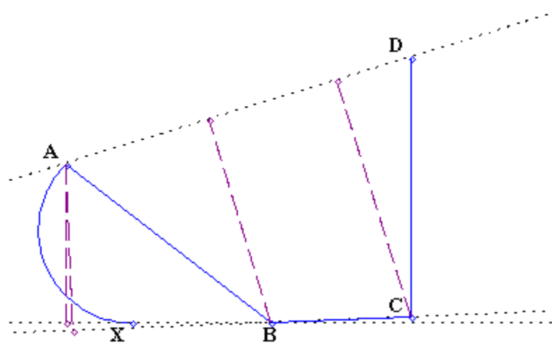


Figure 31

Now we will follow **[1]** more closely.

Proposition 9.3: *A minimal path of unit width is convex.*

Let P be a minimal path of unit width. Then P has an arch of at least unit height. If the arch is the entire path, it is convex and we are finished. If not, select the arch in P of greatest height and orient P with its endpoints on the horizontal axis. Since its endpoints must lie on the convex hull, P is situated entirely above the axis. In Figure 32, the highest arch is $\widehat{ABC}$, X and Y are the endpoints, and R is the rightmost point on the path. The dashed lines indicate sections of the path without specifying their precise shape. L_1 represents the last point on the arch before the crossing to *C*, and L_2 represents the first point on the convex hull after L_1 which is again on path *P*. That is, $\widehat{L_1 C L_2}$ is the next arch in the path.

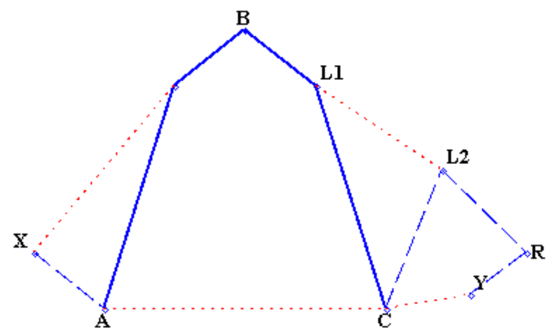


Figure 32

Clearly $\overline{L_1 L_2}$ is not parallel to $\overline{AC}$, since then arc $\widehat{B L_1 C L_2}$ could be replaced by the shorter $\widehat{B C L_2}$ with the same or greater convex hull. So $\overline{L_1 L_2}$ and $\overline{AC}$ intersect at some point *E* (Figure 33). This point must lie to the right of a perpendicular drawn from L_1 to $\overline{AC}$, since otherwise the arc $\widehat{L_1 Y}$ could be replaced by the perpendicular segment dropped from L_1.

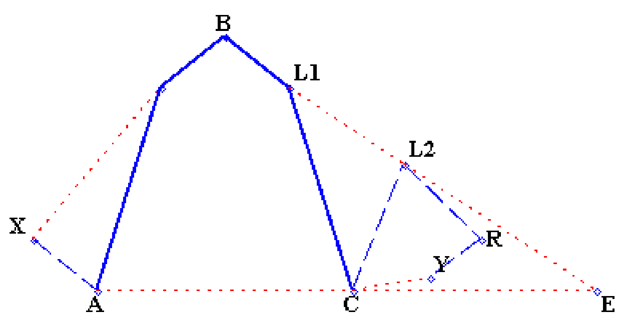


Figure 33

We consider two cases, based on the length of $L_1 L_2$.

Case 1: If $|L_1 L_2| \le |L_1 C|$, we may shorten path *P* and thus show a contradiction as follows. Replace $\widehat{L_1 C L_2}$ with $L_1 L_2$, follow the original path to rightmost point *R*, and drop a perpendicular to point *S* on $\overline{AC}$. Since by convexity *R* lies on or below $\overline{L_1 E}$, $|RS| < |C L_2|$, so the new path must be shorter (Figure 34).

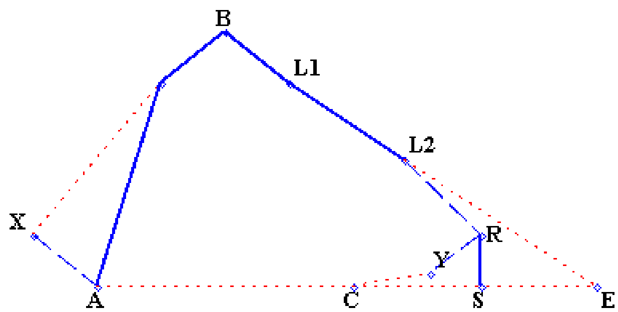


Figure 34

Case 2: (Here we deviate slightly in construction from **[1]** but reach the same objective, namely to show that $m\sphericalangle CEL_1 < 45^\circ$. Then we return to their proof to show that the resulting arc cannot be minimal.) Suppose that $|L_1L_2| > |L_1C|$. Then there must exist a point C' on L_1L_2 with $|L_1C'| = |L_1C|$. (Figure 35)

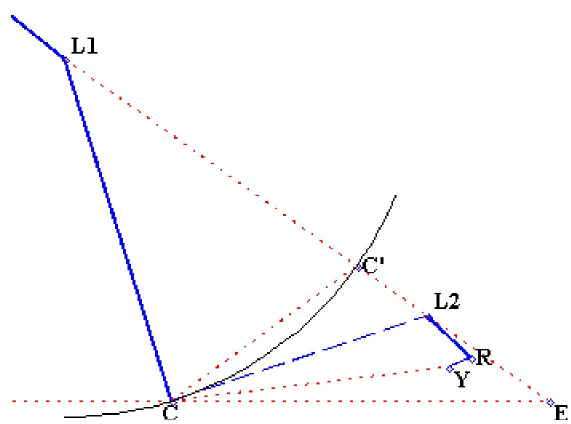


Figure 35

As an exterior angle, $m\sphericalangle CC'L_1 = m\sphericalangle ECC' + m\sphericalangle CEC'$, and since triangle $L_1C'C$ is isosceles, $m\sphericalangle CC'L_1 < 90^\circ$ and $m\sphericalangle CC'E > 90^\circ$. Suppose $\ell(\widetilde{CL_2}) \geq |C'E|$. Then $\ell(\widetilde{L_1CL_2}) = |L_1C| + \ell(\widetilde{CL_2}) = |L_1C'| + \ell(\widetilde{CL_2}) \geq |L_1C'| + |C'E| = |L_1E|$, and we can replace $\widetilde{L_1Y}$ by line segment L_1E to produce a shorter path with a greater convex hull. But then P is not minimal, so we must have $\ell(\widetilde{CL_2}) < |C'E|$. Then $|CC'| < |CL_2| \leq \ell(\widetilde{CL_2}) < |C'E|$, so $m\sphericalangle CEC' < m\sphericalangle ECC'$, and thus $m\sphericalangle CEC' = m\sphericalangle CEL_1 < 45^\circ$.

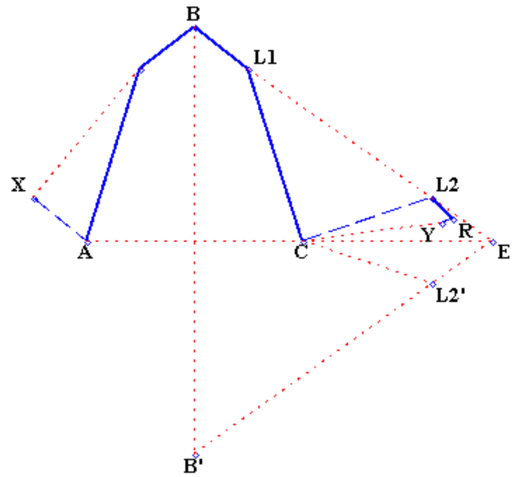


Figure 36

Next we reflect B and L_2 about the horizontal axis to produce points B' and L_2'. Let $r = |BE| = |B'E|$, $h = \frac{|BB'|}{2}$, and $\alpha = m\sphericalangle AEB = m\sphericalangle AEB'$. Let z denote the height of B above $\overline{B'E}$. Then the area of triangle BEB' can be alternately given as $\frac{rz}{2}$ and $hr\cos\alpha$. Equating and solving, we find $z = 2h\cos\alpha$. By inspection, $z \le |BC| + |CL_2'| = |BC| + |CL_2| \le \ell(\widehat{BCL_2})$. Since $h \ge 1$, $z = 2h\cos\alpha \ge 2\cos\alpha$, so $\ell(\widehat{BCL_2}) \ge 2\cos\alpha$. Since $0 < \alpha = m\sphericalangle AEB < 45^\circ$, $\ell(\widehat{BCL_2}) \ge 2\cos(45^\circ) = \sqrt{2}$. But $\ell(\widehat{XAB}) \ge 1$, so $\ell(\widehat{XY}) \ge 1 + \sqrt{2} > 2.414$. This cannot be optimal since we already have a path with length less than 2.3.

Therefore – flourish of trumpets – a minimal path of unit width must be convex. Since we have shown that the Zalgaller path is minimal among convex paths, it is minimal among all paths. We have – thanks to Zalgaller, Schaer, Klötzler, Pickenhain, Adhikari, Pitman, et al. – an optimal escape path from an infinite strip.

- Implications -

Proposition 10 ***(The Zalgaller Upper Bound)***: If a forest F has finite width $w(F)$, then $\ell(F) \le \zeta \cdot w(F)$.

If $w = w(F)$ is the width of forest F, it is contained in an infinite strip of width w, which has escape length $w \cdot \zeta$. So $\ell(F) \le \zeta \cdot w(F)$. Thus $\mathcal{Z}$ provides an upper bound on the escape length of all bounded and some unbounded forests.
One might suspect that the Zalgaller path can be used for other forests. For instance, when a rectangular forest has a very high width-to-length ratio, we are faced with what might as well be an infinite strip, and Zalgaller provides our solution. We should at least examine $\mathcal{Z}$ as a possible solution for any sufficiently “skinny” forest.

The Zalgaller Rectangle

Rectangles form an interesting study, provided in **[5]**. Consider a rectangle *ABCD* with longer sides *AB* and *DC* of length ρ and shorter sides *AD* and *BC* of unit length. As long as $m\sphericalangle ABD > 30^\circ$, the rectangle is fat and so a line segment of length $\sqrt{1+\rho^2}$ is an optimal escape path. This occurs for $1 \le \rho \le \sqrt{3}$. For some sufficiently large value of ρ, we suspect that Zalgaller's path $\mathcal{Z}$ will be an optimal escape. But at what point does $\mathcal{Z}$ become optimal, and what happens between the two solutions? The surprising answer is that there is no "between"!

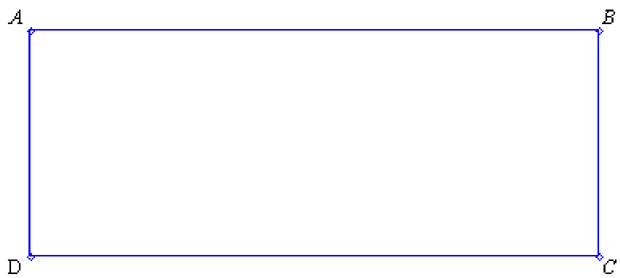


Figure 37

Proposition 11: An optimal escape from a rectangular forest is a diameter path or a Zalgaller path.

Let $d = \sqrt{(1+\rho^2)}$ be the diameter of a $\rho \times 1$ rectangle, and let $\zeta = \ell(\mathcal{Z})$ be the length of the Zalgaller path. Suppose $d < \zeta$, and choose some path *P* of length $L < d$. Certainly, *P* has width less than one and can be positioned between *AB* and *DC*. If *P* intersects both *AD* and *BC* in this orientation but does not intersect *AB* or *DC*, it can also be rotated so that *P* is still contained between but intersects both *AB* and *DC*. If at this point we cannot move *P* horizontally to fit within the rectangle, it must be true that *P* intersects each of the sides at least once. Then regardless of the order in which contact is made, length *L* must be at least that of the diagonal – a contradiction. Therefore, for $d < \zeta$ and escape path *P*, we must have $\ell(P) \ge d$. Since a segment of length *d* is, in fact, the only escape path of length *d*, it is optimal.

When $d > \zeta$, a path of length $L < \zeta < d$ fails as above to escape through sides *AD* and *BC* and cannot escape based solely on *AB* and *DC*, so an escape path must necessarily have $L \ge \zeta$. Since $\mathcal{Z}$ is the unique minimal path of unit width, it is the unique optimal solution when $d > \zeta$.

This leaves us with the case where $d = \zeta$. Here the minimal length is also ζ, but of course that can be accomplished either by a line segment or by $\mathcal{Z}$, and these are the only optimal solutions. This leads to the interesting situation where, as *r* varies continuously, there is an abrupt change in shape of the optimal solution. At the point of change the length of the optimal path ceases to increase with *r* and becomes constant.

Definition: A *Zalgaller rectangle* is a rectangle with diagonal-to-width ratio equal to ζ.

Note that a Zalgaller rectangle has a relatively low length-to-width ratio of $\beta_\zeta = \sqrt{\zeta^2 - 1} \approx 2.0471$. This suggests two results:

Proposition 12 *(The Zalgaller Lower Bound)*: A lower bound for the escape length of a bounded forest *F* is given by $\zeta \cdot w(R) \leq \ell(F)$ where *R* is the largest Zalgaller rectangle covered by *F*.

Together with our general result, this yields $\zeta \cdot w(R) \leq \ell(F) \leq \zeta \cdot w(F)$.

Corollary 13: Call a forest *F* "skinny" if a Zalgaller rectangle *R* can be contained in *F* such that the two longer sides of *R* lie on the boundary of *F*. A Zalgaller path is an optimal escape from a skinny forest.

Convexity requires that $w(R) = w(F)$, so $\zeta \cdot w(F) \leq \ell(F) \leq \zeta \cdot w(F)$. Unfortunately, the skinny forests do not form a family of very great interest.

The Isosceles Triangle: Let forest *F* be bounded by an isosceles triangle ΔABC with base BC and legs $AB \cong AC$, such that the base length $|BC| = \rho$ and *A* has unit height above BC. That is, $\sqrt{|AB|^2 - \left(\frac{\rho}{2}\right)^2} = 1$. Immediately, we have two upper bounds on the escape length of the forest. Since the longest side of a triangle is its diameter, for $0 < m\measuredangle ABC \leq 60^\circ$, $Diam(\Delta ABC) = \rho$ and for $60^\circ < m\measuredangle ABC < 90^\circ$, $Diam(\Delta ABC) = |AB|$, so $\ell(F) \leq max\{\rho, |AB|\}$. But the width of the forest is no greater than its altitude above BC, so $\ell(F) \leq \zeta$.

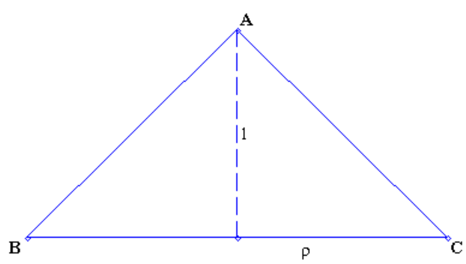


Figure 38

Now if we embed a Zalgaller rectangle along the base (as shown), with height *w* and length $v = w\beta_\zeta$, we see that $\frac{\rho}{2} = \frac{\rho/2}{1} = \frac{\rho/2 - v/2}{w} = \frac{\rho - v}{2w} = \frac{\rho - \beta_\zeta}{2w}$. Then $w\rho = \rho - \beta_\zeta$, and $w = \frac{\rho}{\rho + \beta_\zeta}$. Thus, our embedded rectangle has an escape length of $\frac{\zeta\rho}{\rho + \beta_\zeta}$, which provides a lower bound for the escape length of *F*. Combining our bounds, we have $\zeta \frac{\rho}{\rho + \beta_\zeta} \leq \ell(F) \leq \zeta$.

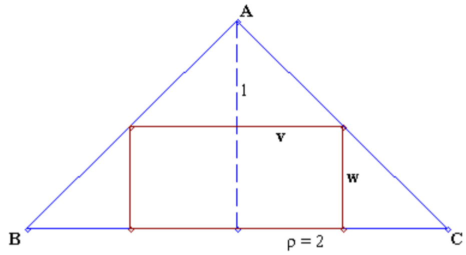


Figure 39

Clearly, for sufficiently large ρ, $\ell(F)$ can be made arbitrarily close to ζ. For example, with a base angle $m\measuredangle BCA \approx 6.2^\circ$ and $\rho \approx 18.43$, $0.9\zeta \le \ell(F) \le \zeta$. And at $m\measuredangle BCA \approx 0.56^\circ$ and $\rho \approx 203$, $0.99\zeta \le \ell(F) \le \zeta$. Thus we suspect that $\mathcal{Z}$ (or something very like it) is optimal for ρ greater than some ρ_0.

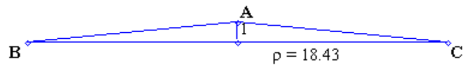


Figure 40

The Ellipse: Let forest *F* be bounded by an ellipse with minor axis of unit length and major axis of length ρ. Embed a Zalgaller rectangle in *F* as shown, and let *d* be the distance from the center to a vertex of the rectangle.

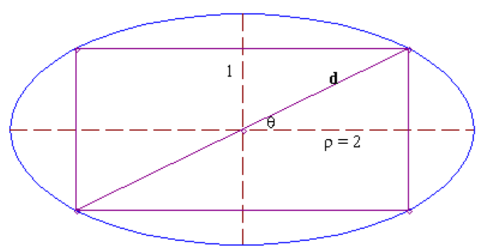


Figure 41

Since the proportions of the Zalgaller rectangle are fixed, the angle of the diagonal to the horizontal axis of the ellipse is given by $\theta = \tan^{-1}\dfrac{1}{\beta_\zeta}$ and the coordinates of the upper-righthand vertex are $(x, y) = (d\cos\theta, d\sin\theta)$. We may write the equation of the ellipse as $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ where $a = \dfrac{\rho}{2}$ and $b = \dfrac{1}{2}$. Thus $\dfrac{d^2\cos^2\theta}{(\rho/2)^2}+\dfrac{d^2\sin^2\theta}{(1/2)^2} = 1 \quad \Rightarrow \quad d^2\left[\left(\dfrac{\cos\theta}{2}\right)^2+\left(\dfrac{\rho\sin\theta}{2}\right)^2\right] = \dfrac{\rho^2}{2^4} \quad \Rightarrow$

$d^2\left(\cos^2\theta+\rho^2\sin^2\theta\right) = \dfrac{\rho^2}{4} \quad \Rightarrow \quad d = \dfrac{\rho}{2\sqrt{\cos^2\theta+\rho^2\sin^2\theta}}$. Since the escape length of the rectangle is its diagonal, $2d = \dfrac{\rho}{\sqrt{\cos^2\theta+\rho^2\sin^2\theta}} \le \ell(F) \le \zeta$. At $\rho \approx 4.23$,

$0.90\,\zeta \le \ell(F) \le \zeta$, and at $\rho \approx 14.37$, $0.99\,\zeta \le \ell(F) \le \zeta$, a considerably more rapid convergence than for the isosceles triangle. Again, we suspect that something like Zalgaller is the optimal escape path for sufficiently elongated ellipses.

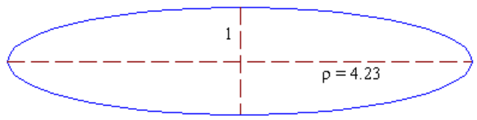


Figure 42

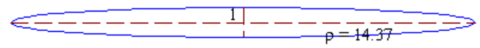


Figure 43

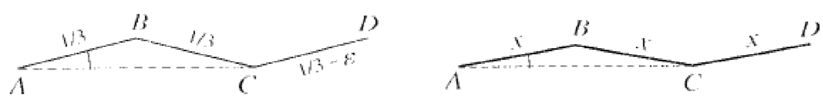
The Besicovitch Solution

Last of the Regular Polygons (Besicovitch, 1965)

With the "Fat Forest" result, we have optimal escape paths for every regular polygonal forest above the triangle. In each of these cases, the solution is a straight line. Perhaps the solution for an equilateral triangle is also linear. After all, the fat rhombus is made up of two equilateral triangles. And the equilateral triangle has the feature that its diameter is simply its side length. And if wishes were horses....

In 2004, Finch and Wetzel **[5]** reported the status as follows:

> Gross **[7]**, however, observed that for sufficiently small ε the path pictured in Figure 5a, with $\measuredangle CAB = 15^\circ$ and $CD = 1/3 - \varepsilon$, is an escape path for the equilateral triangle of unit side, and its length is less than 1. (It is easy to see than any ε with $0 < \varepsilon < 0.013$ works.) Prompted by an equivalent question posed by Graham in 1963, Besicovitch **[3]** found the escape path of length $3\sqrt{21}/14 \approx 0.981981$ pictured in Figure 5b, where $\measuredangle CAB = \arcsin\left(1/\sqrt{28}\right) \approx 10.9^\circ$ and $x = \sqrt{3/28}$. (He obtained his result by solving an optimizing equation numerically; the radical expressions were found by Steven Knox in 1994.) Besicovitch conjectured that this path is the shortest. Although this conjecture is likely to be correct, little progress has been made toward its proof.



(a) The Gross path. (b) The Besicovitch path.

Figure 5. Two zigzag paths, both with $CD \parallel AB$.

Figure 44: From Finch and Wetzel ***[5]***

Just two years later, Coulton and Movshovich **[4]** published their proof of the conjecture, closing the question of the regular polygonal forests.

However, as this paper was begun, the original Besicovitch article was not easily available, Knox's result – "just an elementary calculus problem" done while a grad assistant to Wetzel **[16]** – was never published, and Coulton and Movshovich were not yet known to the author. These lacunae seemed the perfect excuse for a little exploration. So let us consider how one might arrive at Knox's solution, followed by a review of the Coulton and Movshovich proof. (A numerical analysis, probably resembling that of Besicovitch, was also done but doesn't contribute to greater understanding.)

It seems reasonable to assume that the convex hull of any escape path for the equilateral triangle must contact each of the sides at least once, quite possibly utilizing one of the vertices. Following Gross' lead, we consider particularly paths with a quadrilateral convex hull, anticipating a zigzag path of two sides and a diagonal.

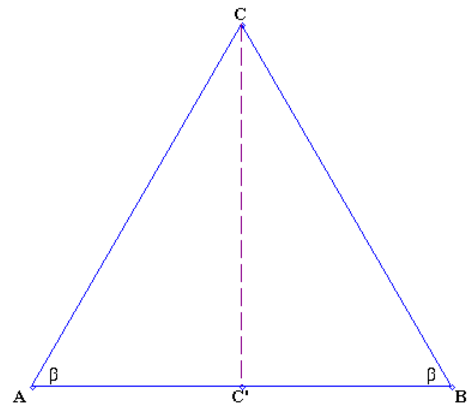


Figure 45

Consider an isosceles triangle *ABC*, where *ABC* has unit base $|AB| = 1$, height of *C* above *AB* of $|CC'| = \kappa$, and base angles $\sphericalangle ABC$ and $\sphericalangle BAC$ with $\beta = m\sphericalangle ABC = m\sphericalangle BAC \leq 60^\circ$.

Note that $\tan\beta = \dfrac{\kappa}{1/2} = 2\kappa$.

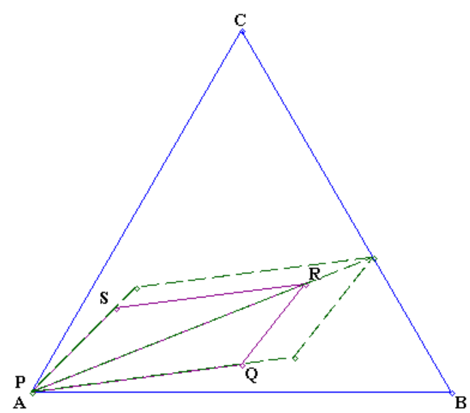


Figure 46

Next let $PQRS$ be a convex quadrilateral hull lying inside ΔABC. Suppose that vertex *P* of the hull and vertex *A* of the triangle are coincident. If point *R* on the hull cannot reach segment *BC* by rotation, we definitely do not have an escape path and can scale the hull larger (Figure 46).
On the other hand, if *R* lies on *BC*, but both *S* and *Q* lie in the interior of the triangle, the hull can be rotated such that the path does not escape the triangle. Inspection suggests that the more we can rotate toward one of the sides, the closer vertex *R* comes to either *B* or *C*. Thus, the worst case positioning of a quadrilateral hull with one vertex at *A* will occur when the side of the quadrilateral forming the shallowest angle to the diagonal lies along one side of the triangle. In the case of a parallelogram, this can be found simply by placing one of the longer sides of the hull along a side of the triangle.

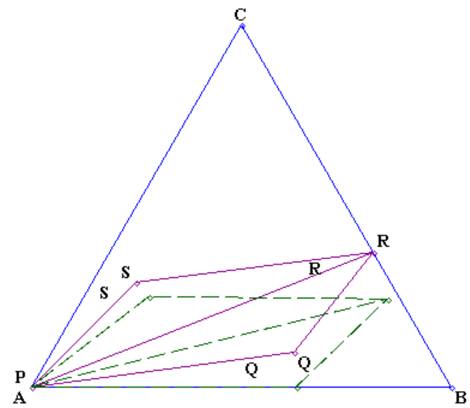


Figure 47

If our triangle is equilateral, choice of side is irrelevant. Otherwise the long side of the hull should lie along the base of the triangle. (If we consider isosceles triangles with base angle larger than 60°, point *P* will need to be coincident with vertex *C* instead of *A*.)

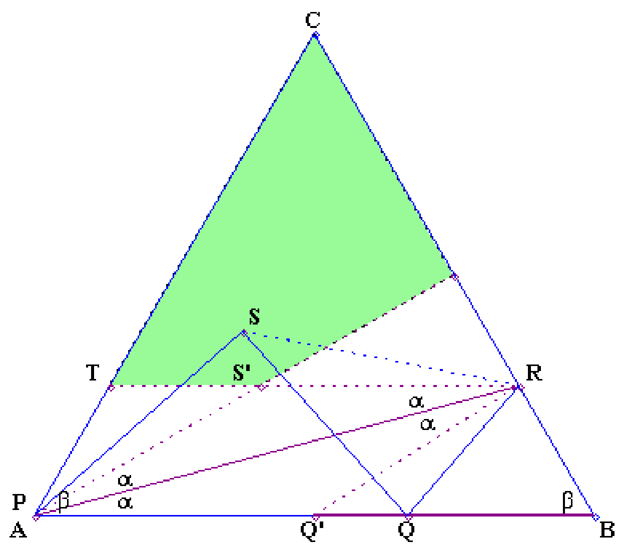


Figure 48

As a result, our goal is effectively to find point *Q* on *AB*, point *R* on *BC*, and point *S* somewhere above *PR* as in Figure 48 such that $\alpha = \measuredangle QPR$ is at least as small as $\measuredangle SPR$, $\measuredangle QRP$, and $\measuredangle SRP$. Such a path should definitely be an escape path. And since we intend to establish the convex hull by a zigzag path, we need to find such a set of points that minimizes $|PS|+|SQ|+|QR|$.

We begin by choosing an arbitrary location for *R* on the lower half of *BC*, and considering the possible locations for *Q* and *S*. Since $m\measuredangle RPS \geq \alpha$ and $m\measuredangle PRS \geq \alpha$, *S* must lie on or above both $\overline{TR}$ and $\overline{PS'}$, where $\overline{TR} \parallel \overline{AB}$ and *S′* lies on $\overline{TR}$ with $m\measuredangle S'PR = \alpha$. Since $m\measuredangle QRP \geq \alpha$, *Q* must lie on segment $Q'B$ where $\overline{Q'R} \parallel \overline{PS'}$.

Consider first the placement of *S*. If *S* lies above both $\overline{TR}$ and $\overline{PS'}$ as in the figure, dropping *S* vertically to the upper of the two lines must decrease the distance $|PS|+|SQ|$ for any fixed *Q*.

And if S lies on $\overline{PS'}$ above $\overline{TR}$, $|PS|+|SQ| = |PS'|+|S'S|+|SQ| \geq |PS'|+|S'Q|$ by the Triangle Inequality. Therefore S must lie on segment TS' if $\widehat{PSQR}$ is to be minimal.

Given Q on $Q'B$, what is the preferred location for S on TS'? Instinctively we know that $|PS|+|SQ|$ is minimized when $PS \cong SQ$. The gods favor symmetry. A simple argument from calculus can be made in support of the supposition, since for fixed base b and height h, $\sqrt{x^2+h^2}+\sqrt{(b-x)^2+h^2}$ is minimized when $x=\frac{b}{2}$. As we will see with Coulton and Movshovich, however, a simple "unfolding" argument is sometimes much more straightforward. If we reflect Q about TR to create point Q', then $\ell(\widehat{PSQ}) = \ell(\widehat{PSQ'})$, which is obviously minimized when $\widehat{PSQ'}$ is a line segment with the result that $PS \cong SQ' \cong SQ$.

Just as we would like to have $PS \cong SQ$, we want $SQ \cong QR$ for the same reasons. So if $PS \cong SQ \cong QR$, our placement is optimal, and if not there is a better placement for S and/or Q. But will it always be possible to place S and Q so that the three legs of the path are congruent? With reference to Figure 49, let $v=|PS|$ be the length of each leg in the path, $w=|PR'|$ the

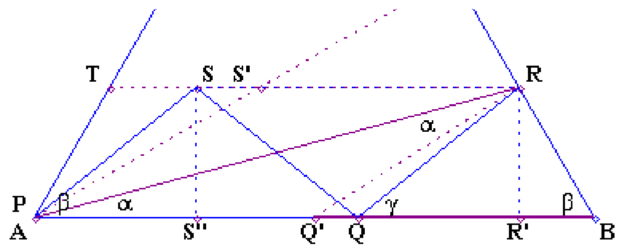


Figure 49

projection of PR onto AB, and $h=|RR'|$ the height of S and R above AB.

Solving $h = w\tan\alpha = (1-w)\tan\beta$ yields $w = \dfrac{\tan\beta}{\tan\alpha+\tan\beta}$, and $h = \dfrac{\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$. Let $m\sphericalangle BQ'R = m\sphericalangle BPS' = 2\alpha$ and $\gamma = m\sphericalangle BQR = m\sphericalangle BPS$. If S lies on TS' and Q on $Q'B$, then $2\alpha \leq \gamma \leq \beta$. In the first quadrant it will be sufficient to verify that $\tan 2\alpha \leq \tan\gamma \leq \beta$.

The three legs of our path are congruent, so $|AS''| = |S''Q| = |QR'| = \frac{w}{3}$ and thus we need $\tan\gamma = \dfrac{h}{w/3} = \dfrac{3h}{w} = 3\tan\alpha \geq \tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$. Then $\dfrac{2\tan\alpha}{3\tan\alpha} = \dfrac{2}{3} \leq 1-\tan^2\alpha$ which implies that $\tan^2\alpha \leq \frac{1}{3}$ or $\alpha \leq 30^\circ$. So for any $0^\circ < 2\alpha \leq \beta \leq 60^\circ$, there is some γ in $[2\alpha, \beta]$ yielding our zigzag path with congruent legs.

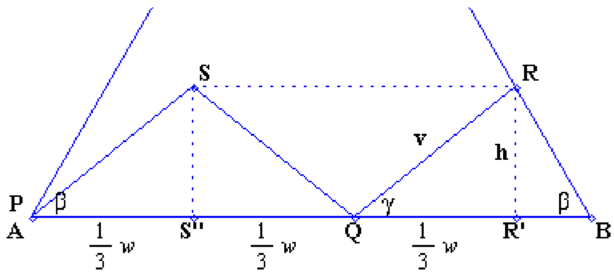


Figure 50

Since our construction is possible for any α, and there is a one-to-one correspondence between the path's γ angle and α, we may concentrate on optimizing path length for $0^\circ < \gamma \le \beta$. As

$h = v \cdot \sin\gamma = (1-w)\tan\beta$ and $\frac{w}{3} = v\cos\gamma$, $\frac{v \cdot \sin\gamma}{\tan\beta} = (1-w)$, and so

$3v \cdot \cos\gamma + v \cdot \frac{\sin\gamma}{\tan\beta} = v\left(3\cos\gamma + \frac{\sin\gamma}{\tan\beta}\right) = 1$ or $v = \left(3\cos\gamma + \frac{\sin\gamma}{\tan\beta}\right)^{-1} = \frac{\tan\beta}{3\cos\gamma\tan\beta + \sin\gamma}$. We minimize $\ell(\overparen{PSQR})$ when v is minimal, which in turn occurs when

$\frac{dv}{d\gamma} = -\left(3\cos\gamma + \frac{\sin\gamma}{\tan\beta}\right)^{-2}\left(-3\sin\gamma + \frac{\cos\alpha}{\tan\beta}\right) = 0$. Since $3\cos\gamma + \frac{\sin\gamma}{\tan\beta}$ is non-zero for

$0^\circ \le \gamma \le 60^\circ$, we may simplify to $\frac{\cos\gamma}{\tan\beta} - 3\sin\gamma = 0 \quad \Rightarrow \quad \cos\gamma - 3\sin\gamma\tan\beta = 0 \quad \Rightarrow$

$\cos\gamma = 3\tan\beta\sin\gamma \quad \Rightarrow \quad \tan\gamma = \frac{1}{3\tan\beta} \quad \Rightarrow \quad \gamma = \tan^{-1}\left(\frac{1}{3\tan\beta}\right)$. Then we can express path

length as the function $P_B(\beta) = \dfrac{3\tan\beta}{\dfrac{9\tan^2\beta}{\sqrt{9\tan^2\beta+1}} + \dfrac{1}{\sqrt{9\tan^2\beta+1}}} = \dfrac{3\tan\beta}{\sqrt{9\tan^2\beta+1}}$. For the

equilateral triangle, $\gamma = \tan^{-1}\left(\frac{1}{3\sqrt{3}}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{9}\right) \approx 10.8934^\circ$ and

$P_B(60^\circ) = \frac{3\sqrt{3}}{\sqrt{28}} = \frac{3\sqrt{21}}{14} \approx 0.98198105061$.

Sorry, folks – all that work bought us a little less than 2% improvement over a straight line solution!

But do we have a problem?

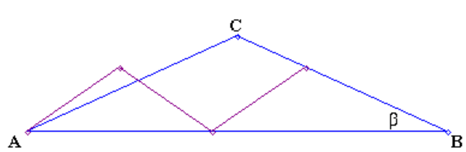


Figure 51

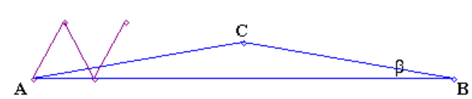


Figure 52

We should note an unexpected result in our calculations. When β is sufficiently small,

$\gamma = \tan^{-1}\left(\dfrac{1}{3\tan\beta}\right)$ will in fact be larger than β. The outcome is a path such as in Figure 51 with $\beta = 25^\circ$ or even Figure 52 with $\beta = 10^\circ$. A quick calculation shows that our path will not fit in a corner of the triangle for $\beta < 30^\circ$. This suggests that a modification of the path might prove a better escape path, but does it invalidate our result?

The generated paths are actually optimal for the zigzag type of path originally specified, if we accept an endpoint lying on the extension of *BC*. If we compare these results with "corrections" that require $\gamma \le \beta$ or even $\gamma \le \beta/2$, a plot of path lengths shows that for $\beta < 30^\circ$ our original result is superior.

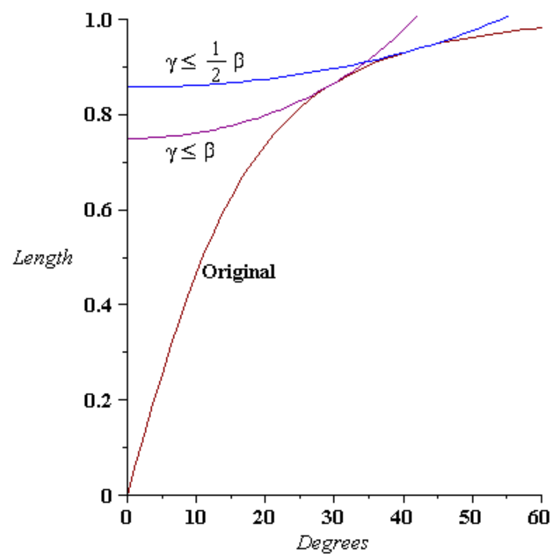


Figure 53

Besicovitch is Optimal (Coulton and Movshovich, 2006)

Finally, in 2006, Patrick Coulton and Yevgenya Movshovich **[4]** established that the Besicovitch conjecture of optimality was correct. Wetzel **[16]** describes their arguments as "not deep, but in the aggregate...elaborate and complicated." As their work is readily available and somewhat lengthy, this paper will confine itself to presenting their results in summary, relying heavily on their original figures. Their own summary is as follows:

> "We will prove that a class of isosceles triangle (called Besicovitch isosceles triangles) are worm-covers. One of these triangles is the Besicovitch equilateral triangle.
>
> In Theorem 5.1 we show that any polygonal arc that does not fit in a Besicovitch isosceles triangle is longer than 1. This means that each Besicovitch isosceles triangle is a worm-cover. The proof employs a rotation of the arc between parallel lines until it assumes a special position. In Sects. 3 and 4 we prove that arcs of this kind are always covered when they are worms." (**[4]**,

p. 80)

Proposition 14: Besicovitch isosceles triangles are unit-covers, and the Besicovitch path forms a unit escape path.

The proof begins by restricting consideration to simple polygonal arcs, claiming "if a convex set covers any simple polygonal unit arc it is a worm-cover." (**[4]**, p. 79.) This claim appeals to a proof-in-summary (contained in an interesting paper on "drapeability" **[10]**) of the assertion found in a 2003 worm problem paper **[12]**.

Specifically, a Besicovitch isosceles triangle T_α is defined as an isosceles triangle "with base angle α satisfying $52.24^\circ \approx \arctan\left(\sqrt{5/3}\right) \le \alpha \le \arctan\left(\sqrt{3}\right) = 60^\circ$ and base $b = \sqrt{1+\frac{1}{9\tan^2(\alpha)}}$."

For our purposes, we will consider a Besicovitch isosceles triangle T_α with base angles of measure α and base length *b* whose base lies on the *x*-axis between the origin and $(b, 0)$. Then the left leg of T_α lies on the line $y = mx,$ where $m = \tan\alpha,$ and the right leg lies on the line $y = m(b-x)$.

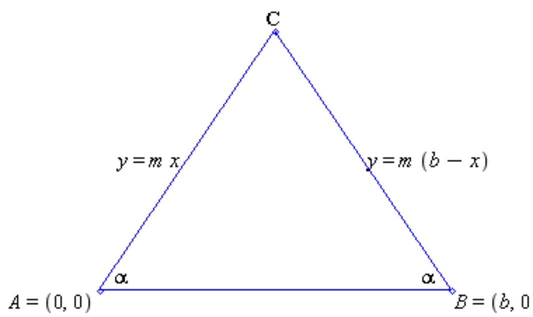


Figure 54

Three-Segment Arcs

We want to show that a 3-segment arc which touches all three sides of the triangle must have at least unit length. We can quickly dispose of *unilateral arcs*, defined as those lying entirely on one side of the line joining the endpoints. A unilateral 3-segment arc can be positioned with its endpoints on the triangle base and clearly must have length greater than the base when the base angle is greater than 45°. Since Besicovitch isosceles triangles have base angle $\alpha \geq \arctan\left(\sqrt{5/3}\right) > 45^\circ$ and base $b = \sqrt{1+\frac{1}{9\tan^2\alpha}} > 1$, every unilateral arc must have length greater than 1.

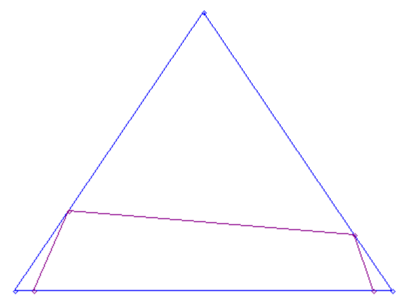


Figure 55

Next we consider certain 3-segment arcs which begin at the origin and end at some point (x, y) on the right leg $y = m(b-x)$. In particular, we define a *symmetric z-arc* to be such an arc where each of the segments is of the same length and has the same altitude *h* above the base.

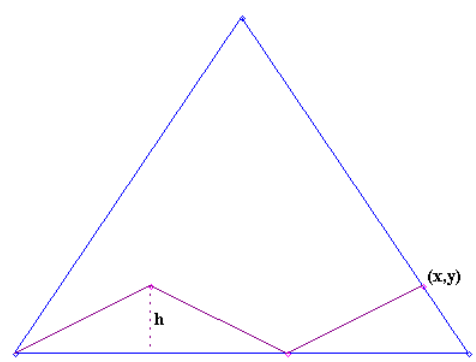


Figure 56

If we "unfold" such a symmetric z-arc, we obtain a line segment between the origin and the point $(x, 3y)$ which lies on the line $y = 3m(b-x)$. Clearly our original arc has minimum length (among symmetric z-arcs) if its unfolded version is orthogonal to $y = 3m(b-x)$. This occurs for symmetric z-arcs of unit length when the triangle base is $b = \sqrt{1+\frac{1}{9\tan^2(\alpha)}}$. Thus, the minimal symmetric z-arc in a Besicovitch isosceles triangle has unit length, and for the equilateral triangle this arc is the Besicovitch path.

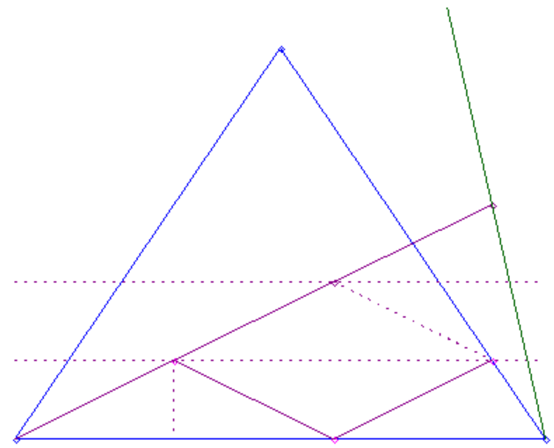


Figure 57

A similar unfolding argument is used to show that a minimal 3-segment arc, can be straightened so that one endpoint lies on the line $y = x \tan(2\alpha)$ and the other is perpendicular to $y = 3m(b-x)$. These two lines are parallel when $\tan(\alpha) = \sqrt{5/3}$ and diverge for greater values of α. Thus, for $\alpha > \arctan(\sqrt{5/3})$ the symmetric z-arc will be the optimal solution, while at the critical angle multiple solutions are conceivable. Therefore, a Besicovitch isosceles triangle will cover all 3-segment arcs of unit length.

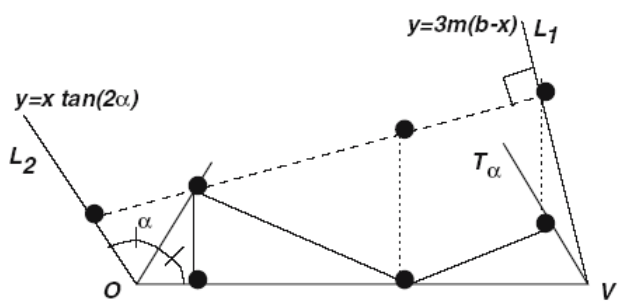


*Figure 58: Coulton and Movshovich **[4]**, page 81*

S-arcs and W-arcs

Coulton and Movshovich define an *s-arc* as one having

> ...an initial point $A = (x_A, y_A)$ on the base; a subsequent point *B* on $y = mx$; a subsequent point *C* of maximum height on $y = h$; a subsequent point *D* on the base; and a terminal point $F = (x_F, y_F)$ on $y = m(b-x)$ such that $y_F \le h$. (**[4]**, p. 82)

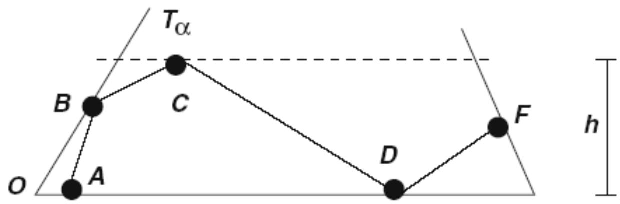


*Figure 59: Coulton and Movshovich **[4]**, p. 82*

They define a *w-arc* as one having

> ...an initial point $A = (x_A, y_A)$ on $y = m\,x$; a subsequent point *B* on the base; a subsequent point *C* on $y = h$; a subsequent point *D* on the base; and a terminal point *F* on $y = m(b - x)$ such that $y_A + y_F \le h$.

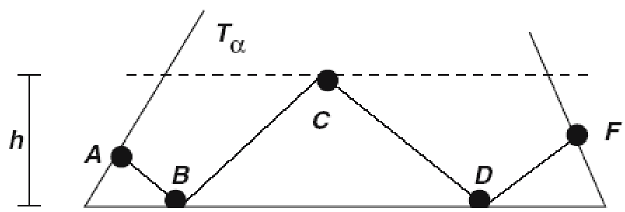


*Figure 60: Coulton and Movshovich **[4]**, page 82*

As their figures demonstrate, an s-arc can be straightened to show that its length is at least the distance between L_1 and L_2, which have a minimum separation of 1.

Fig. 6 An s-arc straightened by reflections

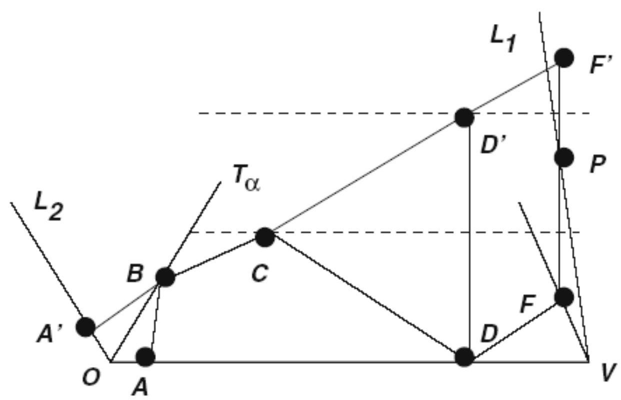


*Figure 61: Coulton and Movshovich **[4]**, page 83*

And a w-arc by unfolding and translation can be shown to correspond to an s-arc.

Fig. 7 Reflection and translate of a w-arc.

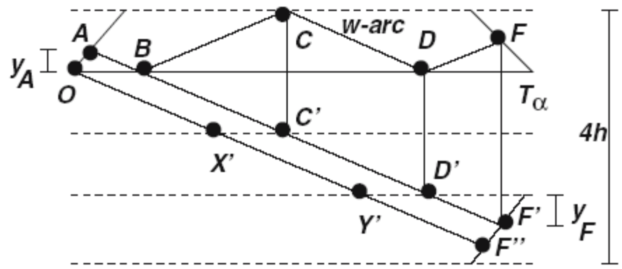


*Figure 62: Coulton and Movshovich **[4]**, page 83*

Fig. 8 A corresponding s-arc

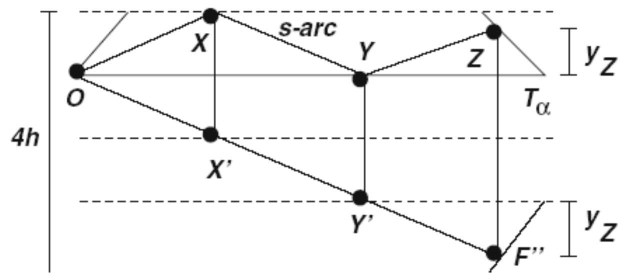


*Figure 63: Coulton and Movshovich **[4]**, page 83*

This leads to the conclusion that every s-arc and every w-arc has length at least 1.

Arcs in Standard Position

Define a $\Lambda - arc$ as lying entirely between the *x*-axis and $y = h$, with consecutive points *A*, *B*, *C*, *D*, *F* such that *A* and *F* are the endpoints, *B* and *D* lie on the *x*-axis, and *C* lies on $y = h$. It is permitted that $A = B$ or $D = F$. A $\Lambda - arc$ is in *standard position* with respect to Besicovitch isosceles triangle T_α if it touches and lies entirely on or to the right of the left side $y = m\,x$ of T_α, and it touches and lies entirely on or to the right of the left side of the inverted copy T_α^* of T_α whose base lies on $y = h$.

Fig. 9 A Λ-arc in standard position

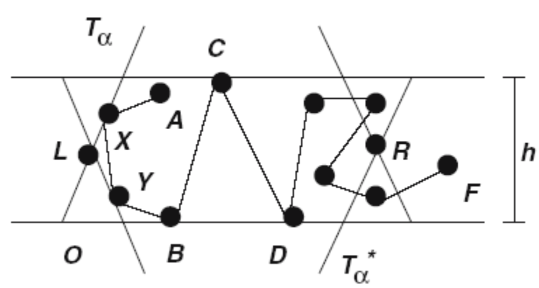


*Figure 64: Coulton and Movshovich **[4]**, page 84*

By careful examination of individual cases, it can be shown that any $\Lambda - arc$ in standard position which is not covered by T_α or T_α^* must have length greater than 1, generally by showing that it can be replaced by a shorter s-arc or w-arc. This leads to the result that T_α must cover every $\Lambda - arc$.

Finally, the authors show that any simple polygonal arc can be rotated so as to eventually take the form of a $\Lambda - arc$. This is then sufficient to conclude that Besicovitch isosceles triangles are worm-covers. Therefore, since the Besicovitch path is an escape path from such triangles (including the equilateral) it must be optimal.

An Exploration

Escape Paths for Isosceles Triangles

We have calculated an optimal zigzag path for isosceles triangles, but have not claimed that the path is optimal among all escape paths. In fact, Coulton and Movshovich **[4]** only show that the Besicovitch path is optimal when the congruent angles measure between about 52.24° and 60°. What might happen with other isosceles triangles?

"Skinny" Triangles

Consider isosceles triangles with base angle $0^\circ < \beta \le 60^\circ$. Again, the "base angle" will refer to the measure of the congruent angles, and "base" to their common side. The height of the triangle is $\frac{\tan\beta}{2}$, so any path of width $\frac{\tan\beta}{2}$ will be an escape path. For small β, a scaled version of the Zalgaller path will have length $P_Z(\beta) = \frac{\zeta \cdot \tan\beta}{2}$ and is certainly a good candidate for best escape path.

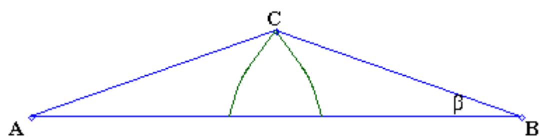


Figure 65

Notice that the scaled Zalgaller path must reach vertex *C* and takes no advantage of the sloping sides nearby. This suggests a third escape path, based on the square, consisting of three congruent legs connected at right angles and thus forming three sides of a square convex hull. If we consider a rectangle of any aspect ratio lying on the base of the triangle and with upper corners on the other sides of the triangle, it is clear that we minimize the length of the three-sided path establishing the rectangular hull by converging to a vertical line when $\beta < 45^\circ$ and to a horizontal line when $\beta > 45^\circ$. However, if we optimize the path beyond the square for one orientation, it will fail to escape the triangle when rotated 90°. Thus for all β the best rectangular hull is square (Figure 66).

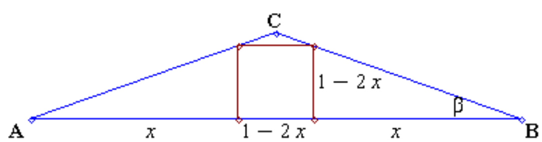


Figure 66

In the configuration shown in *Figure 66*, since $\dfrac{1-2x}{x} = \tan\beta$, we have $x = \dfrac{1}{\tan\beta+2}$, and thus a path length of $P_S(\beta) = 3\left(1-\dfrac{2}{\tan\beta+2}\right) = \dfrac{3\tan\beta}{\tan\beta+2}$. However, we must also ensure that the path is an escape path. It is clear that any placement with one side of the hull lying on the base of the triangle will escape the triangle. It will be sufficient to verify that the path also escapes the triangle when a side of the hull lies on one of the other sides.

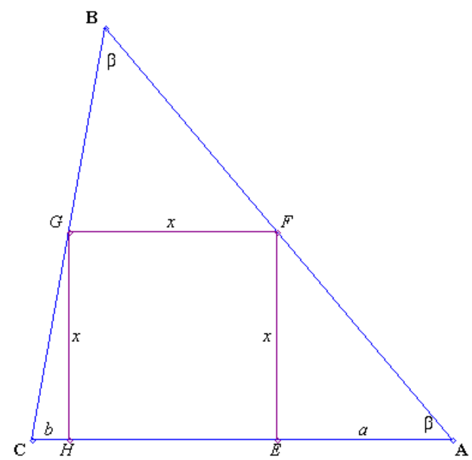


Figure 67

In general, where $\beta = m\sphericalangle BAC = m\sphericalangle ABC$ and $|AB| = 1$, $|AC| = |BC| = \dfrac{1}{2\cos\beta}$. If $\dfrac{\pi}{4} < \beta < \dfrac{\pi}{2}$, as in *Figure 67,* we consider a square lying on one of the congruent sides and sufficiently large to contact the other two sides. Then $\dfrac{x}{a} = \tan\beta \Rightarrow \dfrac{x}{\tan\beta}$, $\dfrac{x}{b} = \tan(\pi-2\beta)$ $= -\tan 2\beta = \dfrac{-2\tan\beta}{1-\tan^2\beta}$, and $x+a+b = \dfrac{1}{2\cos\beta}$. Solving, we find $\dfrac{1}{2\cos\beta} = \dfrac{x(1+\tan^2\beta-2\tan\beta)}{2\tan\beta} \Rightarrow x = \dfrac{\tan\beta}{\cos\beta(1+\tan\beta)^2}$. Since x must be large enough to create an escape in both orientations, we need $x = \max\left\{\dfrac{\tan\beta}{\tan\beta+2}, \dfrac{\tan\beta}{\cos\beta(1+\tan\beta)^2}\right\}$.

However, for $\dfrac{\pi}{4} < \beta < \dfrac{\pi}{3}$, $\dfrac{\sqrt{2}}{2} > \cos\beta > \dfrac{1}{2} \Leftrightarrow 2\cos\beta > 1 \Leftrightarrow$ $2\cos\beta - 2\cos\beta\sin\beta = (1-\sin\beta)2\cos\beta > 1-\sin\beta \Leftrightarrow \sin\beta + 2\cos\beta > 1+2\cos\beta\sin\beta \Leftrightarrow$ $\dfrac{\tan\beta}{\tan\beta+2} = \dfrac{\sin\beta}{\sin\beta+2\cos\beta} < \dfrac{\sin\beta}{1+2\cos\beta\sin\beta} = \dfrac{\sin\beta}{(\cos\beta+\sin\beta)^2} = \dfrac{\sin\beta}{\cos^2\beta(1+\tan\beta)^2} = \dfrac{\tan\beta}{\cos\beta(1+\tan\beta)^2}$.

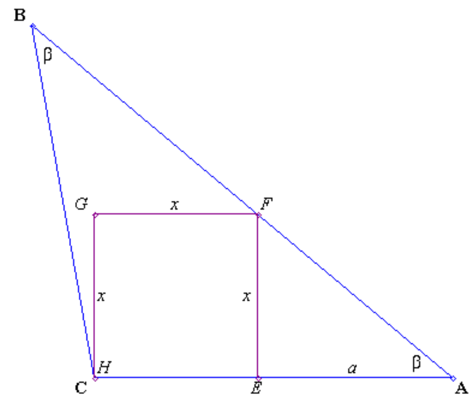


Figure 68

If $0 < \beta \le \frac{\pi}{4}$, as in *Figure 68*, we have $a = \frac{x}{\tan\beta}$ and $x + a = x\left(\frac{1+\tan\beta}{\tan\beta}\right) = \frac{1}{2\cos\beta}$ $\Rightarrow$ $x = \frac{\tan\beta}{2\cos\beta(1+\tan\beta)}$. In all, then, we need $x = \frac{\tan\beta}{\cos\beta(1+\tan\beta)^2}$ when $\beta > \frac{\pi}{4}$, and $x = \max\left\{\frac{\tan\beta}{\tan\beta+2}, \frac{\tan\beta}{2\cos\beta(1+\tan\beta)}\right\}$ when $\beta \le \frac{\pi}{4}$.

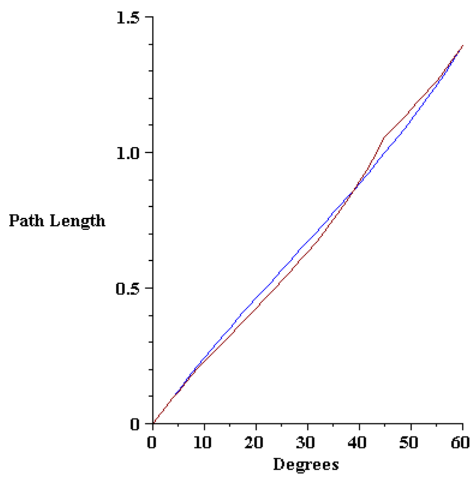


Figure 69

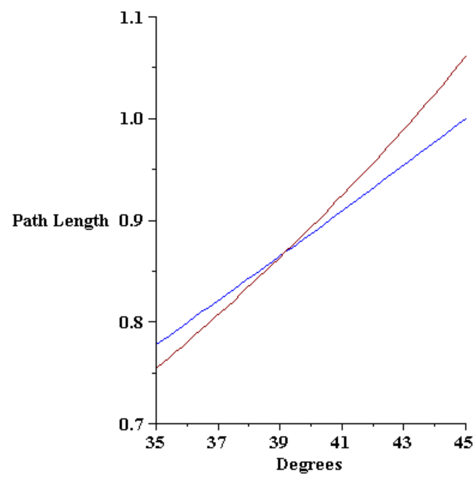


Figure 70

The results are illustrated in *Figures 69 and 70*. The path length is given by $\frac{3\tan\beta}{\tan\beta+2}$ for approximately $0 < \beta \le 39.1^\circ$, $\frac{3\tan\beta}{2\cos\beta(1+\tan\beta)}$ for $39.1^\circ < \beta \le \frac{\pi}{4}$, and $\frac{3\tan\beta}{\cos\beta(1+\tan\beta)^2}$ for $\frac{\pi}{4} < \beta \le \frac{\pi}{3}$.

Among the three path functions – Besicovitch, Zalgaller, and square – for $0^\circ < \beta \le 60^\circ$, we find that the Zalgaller path is minimal for approximately $0^\circ < \beta < 32.36^\circ$, the square path for $32.36^\circ < \beta < 41.34^\circ$, and Besicovitch for $41.34^\circ \le \beta \le 60^\circ$. See *Figures 70 and 71*.

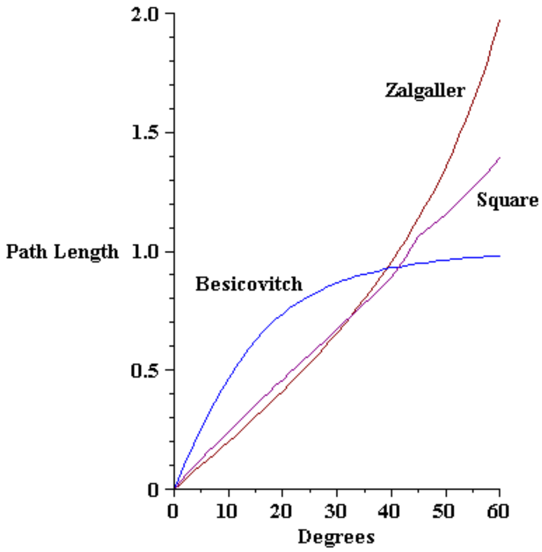


Figure 71

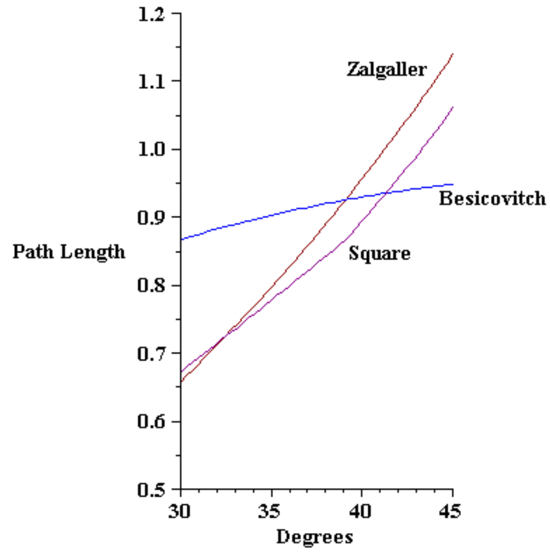


Figure 72

Some numerical experiments indicate that the results for the square can be improved slightly by increasing the lower base to form a path with an isosceles trapezoidal convex hull. This leads to shorter path lengths and a slightly larger range over which the path surpasses Zalgaller and Besicovitch. Which leads us to the following conclusion.

Proposition 15: There exists at least one more type of solution for bounded forests.

We know that for certain isosceles triangles, the optimal escape path – whatever it turns out to be – is not linear, Zalgaller, or Besicovitch.

It is interesting that the Zalgaller path bears some resemblance to a triangle and thus a degenerate trapezoid. Could there be some larger class (perhaps "Zalgalloids") containing the Zalgaller path as one instance, which is optimal for these isosceles triangles?

“Fat” Triangles

For isosceles triangles with $60^\circ < \beta < 90^\circ$, the situation changes slightly. For the Besicovitch path, it is no longer sufficient to place the zigzag across the unit base, since the same path would fit in the apex at *C* and no longer be an escape path. For a corrected Besicovitch length, we need to scale the path length based on the longer side as a base. Since $|AC| = \frac{1}{2\cos\beta}$, our modified function is $Q_B(\beta) = \frac{3\tan\beta}{2\cos\beta\sqrt{9\tan^2\beta+1}}$. And as the path length must traverse a long side of the triangle, we expect increasingly poor results as β grows.

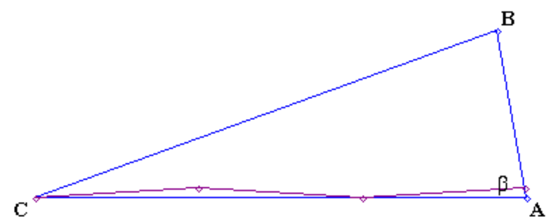


Figure 73

The Zalgaller path requires similar scaling, but varies with $\sin\beta$ rather than the inverse of $\cos\beta$. As a result, the path grows much more slowly and converge to the unit width ζ as β approaches 90°. The height of the triangle measured from its longer bases is $\sin\beta$, so $Q_Z(\beta) \approx 2.278291644 \sin\beta$. It is possible that the path can be scaled down further because of the position with respect to the altitude of the triangle, but this has not been investigated.

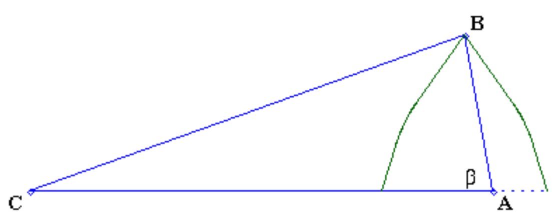


Figure 74

The square path is not affected by the growth of β, so $Q_S(\beta) = P_S(\beta) = \frac{3\tan\beta}{\tan\beta+2}$.

Comparing the three paths for $60^{\circ} < \beta < 90^{\circ}$, we see a reverse of the pattern for the smaller angles. The Besicovitch path is minimal (among the three) for approximately $60^{\circ} < \beta < 75.356^{\circ}$, the square path for $75.356^{\circ} < \beta < 80.487^{\circ}$, and the Zalgaller path for $80.487^{\circ} < \beta \le 90^{\circ}$.

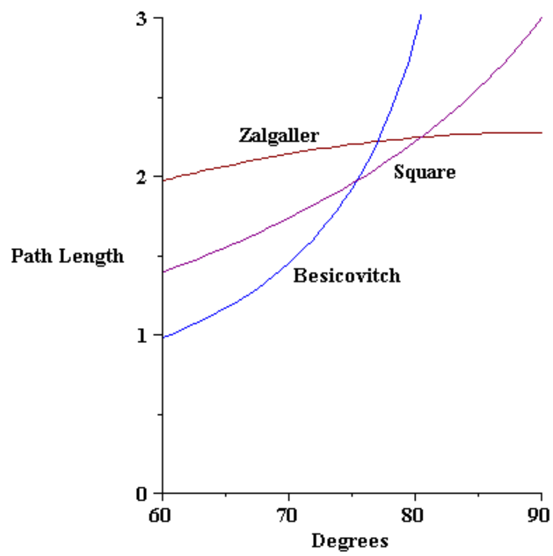


Figure 75

A Brief History of Results

The following sketch of the problems' history is based on Wetzel and Finch's 2004 "Lost in a Forest" **[5]**, and a recent letter from Wetzel in *Geombinatorics* **[15]**.

- 1956 – Bellman **[2]** asks, given a forest of known dimensions, for the shortest escape path. The problem was originally posed for an infinite strip of constant width and for a half-plane with known distance from the boundary.
- 1955 – Gross **[7]** solves the circular disk and comments on other forests. (Clearly Bellman had been talking about the problem before his original publication.) This may have included his observation on the equilateral triangle that foreshadowed the Besicovitch solution.
- 1957 – Isbell **[8]** solves the half-plane.
- 1961 – Zalgaller **[19]** solves the infinite strip. The solution will be rediscovered and the path renamed several times.
- 1965 – Besicovitch **[3]** refines the observation by Gross regarding the equilateral triangle and offers a numerical solution which will eventually prove to be optimal.
- 1968 – Schaer, in a University of Calgary research paper, shows the Zalgaller path is optimal. The path is dubbed the *broadworm*, since it has the greatest possible width-to-length ratio.
- 1973 – Poole and Gerriets show that the solution for a 60-degree rhombus is linear. Their preliminary letter **[13]** is followed by a more thorough proof **[6]** in 1974. This result establishes the class of "fat" forests.
- 1986 – Klötzler and Pickenhain rediscover the Zalgaller path, this time naming it the *universal escape path* (which it clearly is not).
- 1989 – Adhikari and Pitman **[1]** discover yet again the Zalgaller path, descriptively calling it the *caliper*.
- 1994 – Stephen Knox, while a graduate student, provides an algebraic representation for the Besicovitch path.
- 2004 – Finch and Wetzel **[5]** publish an article reviewing results to date.
- 2006 – Coulton and Movshovich **[4]** publish their proof that the Besicovitch path is indeed optimal for equilateral triangles and an additional range of isosceles triangles.

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