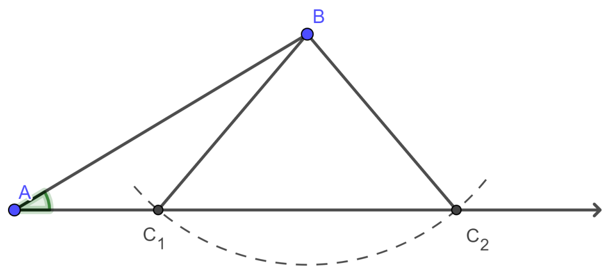
Sometimes SSA - A restriction for SSA congruence

When most students are introduced to triangle congruence, they learn various congruence rules - typically SSS, SAS, and ASA, and possibly AAS and HL. They are carefully instructed that SSA - that is, two sides and a non-included angle are insufficient to guarantee congruence. A simple pendulum example with the swinging third side is enough to make the point.



But are there conditions, other than simply another side or angle congruence, under which SSA is sufficient to guarantee triangle congruence? Two cases look promising.

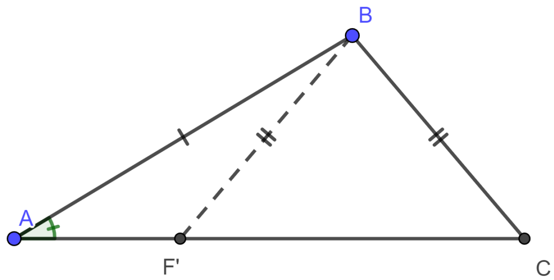
1. If the angle is right or obtuse, the "pendulum" problem appears to be eliminated. In fact, HL (hypotenuse-leg) congruence is a restricted SSA congruence.
2. Likewise, the "pendulum" problem disappears if the side opposite the angle is at least as long as the other given side. Upon examination, the second case includes the first case, so I'll present that first.

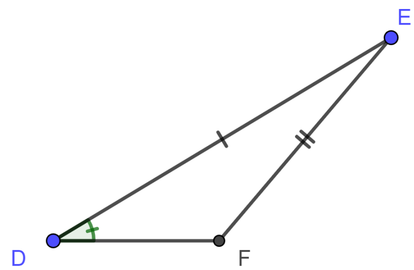
This all seems an obvious observation, but a proof is offered anyway.

Proposition: Let ΔABC and ΔDEF be triangles with $\angle A \cong \angle D$, $AB = DE$, and $BC = EF$. If $BC \geq AB$, then the triangles are congruent.

Proof: Let ΔABC and ΔDEF be triangles with $\angle A \cong \angle D$, $AB = DE$, and $BC = EF$.

Suppose $\Delta ABC \ncong \Delta DEF$. Then $AC \neq DF$. Assume, without loss of generality, that ΔDEF has the shorter third side - that is, $DF < AC$. Then there must be a point F' on $\overline{AC}$ with $AF' = DF$.





By SAS, $\Delta ABF' \cong \Delta DEF$ and so $BF' = EF = BC$. Then triangle $\Delta F'BC$ is isosceles, so $\angle ACB$ must be an acute angle and $\angle AF'B$ must be obtuse. Since the larger side must be opposite the larger angle, $AB > BF' = BC$.

Thus, if $\Delta ABC \ncong \Delta DEF$, we must have $AB > BC$. And by contraposition, if $BC \geq AB$, then $\Delta ABC \cong \Delta DEF$.

(Note that the condition is sufficient but not necessary. If $BC < AB$, the triangles could still be congruent, but we would need further information to prove it.)

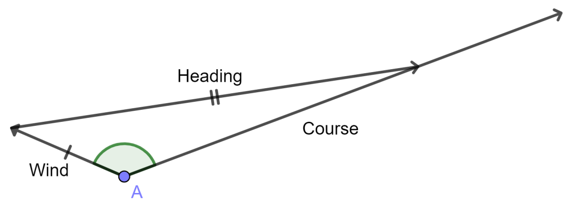
Corollary: Given triangles ΔABC and ΔDEF as above, if $\angle A$ is right or obtuse, then we must have $BC > AB$ and so $\Delta ABC \cong \Delta DEF$.

This observation arises from thinking about applications of triangles and triangle congruences. One familiar example comes from plotting a course in aviation. Given the desired course direction, the direction and magnitude of the wind, and the cruising speed of the airplane, what heading (direction) should the pilot fly and what will be the resulting ground speed?

This can all be solved with trigonometry, but ruler and protractor suffice.

1. Plot a course line in the desired direction of travel.
2. Draw a vector showing wind direction and speed rooted at the beginning of the course line.

3. Swing the ruler to form a vector of cruising speed magnitude from the end of the wind vector to the course line. The new vector gives the heading and the resulting course line vector gives the resulting ground speed.



The problem of using this example with students, of course, is that the construction is based on SSA congruence but we'd like to believe that the construction can only result in one heading and ground speed. Hence the utility of the SSA restriction. Since few pilots are likely to attempt flight when the wind speed exceeds their cruise speed, the SSA construction is safe.